Queueing Systems

Bo Friis Nielsen[1]

[1]DTU Informatics

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Today:

- ▶ Finite state continuous time Markov chains
- ▶ Queueing Processes
- ▶ M/M/s systems
- ▶ M/G/1 system

After fall break

- ▶ To be announced

Finite Contiunuous Time Markov Chains

$$P_{ij}(t) = \mathbb{P}\{X(t+s) = j | X(s) = i\}$$

(a) $P_{ij}(t) \geq 0$

(b) $\sum_{j=0}^{N} P_{ij}(t) = 1$

(c) $P_{ik}(s+t) = \sum_{j=0}^{N} P_{ij}(s) P_{jk}(t)$

(d) $\lim_{t \to 0+} P_{ij}(t) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

(c) is the Chapman-Kolmogorov equations, in matrix form

$$\boldsymbol{P}(t+s) = \boldsymbol{P}(t)\boldsymbol{P}(s)$$

$$
\begin{aligned}
\boldsymbol{P}'(t) &= \boldsymbol{P}(t)\boldsymbol{A} (\text{ forward equations}) \\
\boldsymbol{P}'(t) &= \boldsymbol{A}\boldsymbol{P}(t) (\text{ backward equations}) \\
\boldsymbol{P} &= e^{\boldsymbol{A}t} = \sum_{n=0}^{\infty} \frac{\boldsymbol{A}^n t^n}{n!} = \boldsymbol{I} + \sum_{n=1}^{\infty} \frac{\boldsymbol{A}^n t^n}{n!}
\end{aligned}
$$

Stationary Distribution

$$\mathbf{0} = \pi \mathbf{A} = (\pi_0, \pi_1, \ldots, \pi_N) \begin{Vmatrix} -q_0 & q_{01} & \ldots & q_{0N} \\ q_{10} & -q_1 & \ldots & q_{1N} \\ \vdots & \vdots & \ldots & \vdots \\ q_{N0} & q_{N1} & \ldots & q_{NN} \end{Vmatrix}$$

elementwise

$$\pi_j q_j = \sum_{i \neq j} \pi_i q_{ij} \text{ with } q_j = \sum_{k \neq j} q_{jk}$$

Infinitesimal Description

$$\mathbb{P}\{X(t+h) = j | X(t) = i\} = q_{ij}h + o(h) \text{ for } i \neq j$$
$$\mathbb{P}\{X(t+h) = i | X(t) = i\} = (1 - q_i h) + o(h)$$

Sojourn Description

1. Embedded Markov chain of state sequences ξ_i has one step transition probabilities $p_{ij} = \frac{q_{ij}}{q_i}$

2. Successive sojourn times S_{ξ_i} are exponentially distributed with mean $\frac{1}{q_{\xi_i}}$.

Two State Markov Chain

$$\mathbf{A} = \begin{Vmatrix} -\alpha & \alpha \\ \beta & -\beta \end{Vmatrix}$$

$$\mathbf{A}^2 = \begin{Vmatrix} \alpha^2 + \alpha\beta & -\alpha^2 - \alpha\beta \\ -\beta^2 - \alpha\beta & \beta^2 + \alpha\beta \end{Vmatrix} = -(\alpha + \beta) \begin{Vmatrix} -\alpha & \alpha \\ \beta & -\beta \end{Vmatrix}$$

Thus

$$\mathbf{A}^n = [-(\alpha + \beta)]^{n-1} \mathbf{A}$$

And

$$\begin{aligned} \mathbf{P}(t) &= \mathbf{I} - \frac{1}{\alpha + \beta} \sum_{n=1}^{\infty} \frac{[-(\alpha + \beta)t]^n}{n!} \mathbf{A} \\ &= \mathbf{I} - \frac{1}{\alpha + \beta} \left(e^{-(\alpha+\beta)t} - 1 \right) \mathbf{A} \\ &= \mathbf{I} + \frac{1}{\alpha + \beta} \mathbf{A} - \frac{1}{\alpha + \beta} \mathbf{A} e^{-(\alpha+\beta)t} \end{aligned}$$

Summary of most Important Results

$$\mathbf{P}(t) = e^{\mathbf{A}t} \to \mathbf{1}\pi \quad \text{for } t \to \infty$$

Under an assumption of irreducibility

Additional Reading

Kai Lai Chung: "Markov Chains with Stationary Transition Probabilities"

Queueing process

1. Input Process
2. Service Process
3. Queue Discipline

Kendall notation: $A/B/s$

1. Number of customers/items in system
2. Throughput
3. Utilization
4. Customer waiting time, probability of being served

Little's Theorem

$$L = \lambda W$$

M/M/1 System

Steady state equations:

$$\pi_k = \lim_{t \to \infty} \mathbb{P}\{X(t) = k\} \text{ for } k = 0, 1, \ldots$$

$$\pi_k \mu = \pi_{k-1}\lambda, \qquad \pi_k = \theta_k \pi_0$$

with

$$\theta_k = \left(\frac{\lambda}{\mu}\right)^k$$

such that

$$\pi_0 \sum_{k=0}^{\infty} \theta_k = 1 \Leftrightarrow \pi_0 = \frac{1}{\sum_{k=0}^{\infty} \theta_k} = \frac{1}{\frac{1}{1-\frac{\lambda}{\mu}}} = 1 - \frac{\lambda}{\mu} \text{ for } \lambda < \mu$$

$$\pi_k = \left(\frac{\lambda}{\mu}\right)^k \left(1 - \frac{\lambda}{\mu}\right) = \rho^k (1 - \rho) \text{ with } \rho = \frac{\lambda}{\mu}$$

A geometric distribution

Total Time in M/M/1 System

$$\mathbb{P}\{T \leq t | n \text{ ahead}\} = \int_0^t \mu \frac{(\mu\tau)^n}{n!} e^{-\mu\tau} d\tau$$

$$
\begin{aligned}
\mathbb{P}\{T \leq t\} &= \sum_{n=0}^{\infty} \mathbb{P}\{T \leq t | n \text{ ahead}\} \mathbb{P}\{n \text{ ahead}\} \\
&= \sum_{n=0}^{\infty} \left[\int_0^t \mu \frac{(\mu\tau)^n}{n!} e^{-\mu\tau} d\tau \right] \left(1 - \frac{\lambda}{\mu} \right) \left(\frac{\lambda}{\mu} \right)^n \\
&= \int_0^t \mu \left(1 - \frac{\lambda}{\mu} \right) e^{-\mu\tau} \sum_{n=0}^{\infty} \frac{(\mu\tau)^n}{n!} \left(\frac{\lambda}{\mu} \right)^n \\
&= \int_0^t (\mu - \lambda) e^{-(\mu-\lambda)\tau} d\tau \\
&= 1 - e^{-(\mu-\lambda)t}
\end{aligned}
$$

Mean performance measures

$$W = \frac{1}{\mu - \lambda}$$

From Littles theorem

$$L = \lambda W = \frac{\lambda}{\mu - \lambda} = \frac{\rho}{1 - \rho}$$

which we could also have gotten directly from the distribution π_k

M/M/1 Busy Period

$$\mathbb{E}[I_1] = \frac{1}{\lambda}$$

$$\lim_{t \to \infty} p_0(t) = \pi_0 = \frac{\mathbb{E}[I_1]}{\mathbb{E}[I_1] + \mathbb{E}[B_1]}$$

$$1 - \frac{\lambda}{\mu} = \frac{\frac{1}{\lambda}}{\frac{1}{\lambda} + \mathbb{E}[B_1]}$$

$$\mathbb{E}[B_1] = \frac{1}{\mu - \lambda}$$

M/M/∞ System

$$\lambda_k = \lambda \qquad \mu_k = k\mu$$

$$\pi_k \lambda = \pi_{k+1}(k+1)\mu \qquad \theta_k = \frac{\left(\frac{\lambda}{\mu} \right)^k}{k!}$$

$$\sum_{k=0}^{\infty} \pi_k = \pi_0 \sum_{k=0}^{\infty} \theta_k = \pi_0 \sum_{k=0}^{\infty} \frac{\left(\frac{\lambda}{\mu} \right)^k}{k!} = \pi_0 e^{\frac{\lambda}{\mu}}$$

$$\pi_k = \frac{\left(\frac{\lambda}{\mu} \right)^k}{k!} e^{-\frac{\lambda}{\mu}}$$

a Poisson distribution

M/M/s System

$$\lambda_k = \lambda, \mu_k = \begin{cases} k\mu & \text{for } k < s \\ s\mu & \text{for } s \leq k \end{cases}$$

$$\pi_0 = \left\{ \sum_{j=0}^{s-1} \frac{1}{j!} \left(\frac{\lambda}{\mu}\right)^j + \frac{\left(\frac{\lambda}{\mu}\right)^s}{s! \left(1 - \frac{\lambda}{s\mu}\right)} \right\}$$

$$\pi_k = \begin{cases} \frac{1}{k!} \left(\frac{\lambda}{\mu}\right)^k \pi_0 & \text{for } k = 0, 1, \ldots, s \\ \frac{1}{s!} \left(\frac{\lambda}{\mu}\right)^s \left(\frac{\lambda}{s\mu}\right)^{k-s} \pi_0 & \text{for } k \geq s \end{cases}$$

M/M/s System: Performance Measures

$$
\begin{aligned}
L_0 &= \sum_{j=s}^{\infty} (j-s)\pi_j \\
&= \pi_0 \sum_{k=0}^{\infty} k \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!} \left(\frac{\lambda}{s\mu}\right)^k \\
&= \frac{\pi_0}{s!} \left(\frac{\lambda}{\mu}\right)^s \sum_{k=0}^{\infty} k \left(\frac{\lambda}{s\mu}\right)^k \\
&= \frac{\pi_0}{s!} \left(\frac{\lambda}{\mu}\right)^s \frac{\left(\frac{\lambda}{s\mu}\right)}{\left(1 - \frac{\lambda}{s\mu}\right)^2} \\
W_0 &= \frac{L_0}{\lambda} \qquad W = W_0 + \frac{1}{\mu} \qquad L = \lambda W = L_0 + \frac{\lambda}{\mu}
\end{aligned}
$$

M/G/1 System

Let $Y_1, Y_2, \ldots$ be a sequence of service times with cumulative distribution function $G(y)$, with mean $\nu = \mathbb{E}[Y_k]$

$$\lim_{t \to \infty} p_0(t) = \pi_0 = \frac{\mathbb{E}[I_1]}{\mathbb{E}[I_1] + \mathbb{E}[B_1]}$$

$\mathbb{E}[I_1] = \frac{1}{\lambda}$, A: number of arrivals during service time of first customer of the busy period

$$\mathbb{E}[B_1 | Y_1 = y, A = 0] = y, \qquad \mathbb{E}[B_1 | Y_1 = y, A = 1] = y + \mathbb{E}[B_1]$$

$$\mathbb{E}[B_1 | Y_1 = y, A = 1] = y + n\mathbb{E}[B_1]$$

$$\mathbb{E}[B_1 | Y_1 = y] = \sum_{n=0}^{\infty} (y + n\mathbb{E}[B_1]) \frac{(\lambda y)^n}{n!} e^{-\lambda y} = y + \lambda y \mathbb{E}[B_1]$$

Finally $\mathbb{E}[B_1] = \mathbb{E}[\mathbb{E}[B_1 | Y]] = \nu + \nu\lambda\mathbb{E}[B_1]$ such that

$$\mathbb{E}[B_1] = \frac{\nu}{1 - \nu\lambda} \text{ and } \pi_0 = \frac{\frac{1}{\lambda}}{\frac{1}{\lambda} + \frac{\nu}{1-\nu\lambda}} = 1 - \nu\lambda \text{ for } \lambda\nu < 1$$

M/G/1 Embedded Markov chain at departures

$$
\begin{aligned}
X_{n+1} &= \begin{cases} X_n - 1 + A_n & \text{for } X_n \geq 1 \\ A_n & \text{for } X_n = 0 \end{cases} \\
&= (X_n - 1)^+ A_n \\
\alpha_k &= \mathbb{P}\{A_n = k\} = \int_0^{\infty} \frac{(\lambda y)^k}{k!} e^{-\lambda y} \mathrm{d}G(y) \\
P_{ij} &= \begin{cases} \alpha_{j-i-1} & \text{for } j \geq i - 1 \geq 0 \\ \alpha_j & \text{for } i = 0 \end{cases}
\end{aligned}
$$

Mean Queue Length in Equilibrium

$$\lim_{n\to\infty} \mathbb{P}\{X_n = k\} = \pi_k, \qquad \lim_{n\to\infty} \mathbb{E}[X_n] = L$$

$$X_{n+1} = X_n - \delta + A_n, \qquad \mathbb{E}[X_{n+1}] = \mathbb{E}[X_n] - \mathbb{E}[\delta] + \mathbb{E}[A_n]$$

Leading to $\mathbb{E}[A_n] = \mathbb{E}[\delta] = 1 - \pi_0 = \lambda\nu$
Now by Squaring

$$
\begin{aligned}
X_{n+1}^2 &= X_n^2 + \delta^2 + A_n^2 - 2X_n\delta + 2A_n(X_n - \delta) \\
\mathbb{E}\left[X_{n+1}^2\right] &= \mathbb{E}\left[X_n^2\right] + \mathbb{E}[\delta] + \mathbb{E}\left[A_n^2\right] - 2\mathbb{E}[X_n] + 2\mathbb{E}[A_n]\mathbb{E}[X_n - \delta] \\
0 &= \lambda\nu + \mathbb{E}\left[A_n^2\right] - 2L + 2\lambda\nu(L - \lambda\nu) \\
L &= \frac{\lambda\nu + \mathbb{E}\left[A_n^2\right] - 2(\lambda\nu)^2}{2(1 - \lambda\nu)}
\end{aligned}
$$

M/G/1 Calculation of $\mathbb{E}\left[A_n^2\right]$

$$\mathbb{E}\left[A_n^2 | Y = y\right] = (\lambda y + (\lambda y)^2)$$

$$\mathbb{E}\left[A_n^2\right] = \int_0^\infty \left(\lambda y + (\lambda y)^2\right) dG(y) = \lambda\nu + \lambda^2(\nu^2 + \tau^2)$$

with $\tau^2 = \mathbb{V}\text{ar}(Y)$. Finally, inserting and rearranging

$$L = \frac{\lambda\nu + \lambda\nu + \lambda^2(\nu^2 + \tau^2) - 2(\lambda\nu)^2}{2(1 - \lambda\nu)} = \rho + \frac{\lambda^2\tau^2 + \rho^2}{2(1 - \lambda\nu)}$$

The distributions of $X(t)$ and X_n are identical

M/G/∞ system

$$A_t = \{(w, v) : 0 \leq w \leq t \text{ and } v > t - w\}$$

$$
\begin{aligned}
\mu(A_t) &= \int_0^t \left\{\int_{t-w}^\infty dG(y)\right\} dw \\
&= \lambda \int_0^t [1 - G(t - w)] dw \\
&= \lambda \int_0^t [1 - G(x)] dx
\end{aligned}
$$

$$\mathbb{P}\{X(t) = k\} = \frac{(\lambda\mu(A_t))^k}{k!} e^{-\lambda\mu(A_t)} \to \frac{(\lambda\nu)^k}{k!} e^{-\lambda\nu} \text{ as } t \to \infty$$