Queueing Systems

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Queueing Systems

Today:

- Finite state continuous time Markov chains
- Queueing Processes
- M/M/s systems
- M/G/1 system

After fall break

- To be announced

Finite Contiunuous Time Markov Chains

$$P_{ij}(t) = \mathbb{P}\{X(t+s) = j | X(s) = i\}$$

(a) $P_{ij}(t) \geq 0$

(b) $\sum_{j=0}^{N} P_{ij}(t) = 1$

(c) $P_{ik}(s+t) = \sum_{j=0}^{N} P_{ij}(s) P_{jk}(t)$

(d) $\lim_{t \to 0+} P_{ij}(t) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

(c) is the Chapman-Kolmogorov equations, in matrix form

$$\boldsymbol{P}(t+s) = \boldsymbol{P}(t)\boldsymbol{P}(s)$$

$$
\begin{aligned}
\boldsymbol{P}'(t) &= \boldsymbol{P}(t)\boldsymbol{A}(\text{ forward equations}) \\
\boldsymbol{P}'(t) &= \boldsymbol{A}\boldsymbol{P}(t)(\text{ backward equations}) \\
\boldsymbol{P} &= e^{\boldsymbol{A}t} = \sum_{n=0}^{\infty} \frac{\boldsymbol{A}^n t^n}{n!} = \boldsymbol{I} + \sum_{n=1}^{\infty} \frac{\boldsymbol{A}^n t^n}{n!}
\end{aligned}
$$

Stationary Distribution

$$\mathbf{0} = \boldsymbol{\pi A} = (\pi_0, \pi_1, \ldots, \pi_N) \left\| \begin{matrix} -q_0 & q_{01} & \cdots & q_{0N} \\ q_{10} & -q_1 & \cdots & q_{1N} \\ \vdots & \vdots & \cdots & \vdots \\ q_{N0} & q_{N1} & \cdots & q_{NN} \end{matrix} \right\|$$

elementwise

$$\pi_j q_j = \sum_{i \neq j} \pi_i q_{ij} \text{ with } q_j = \sum_{k \neq j} q_{jk}$$

Infinitesimal Description

$$\begin{aligned}\mathbb{P}\{X(t+h)=j|X(t)=i\} &= q_{ij}h+o(h) \text{ for } i\neq j \\ \mathbb{P}\{X(t+h)=i|X(t)=i\} &= (1-q_ih)+o(h)\end{aligned}$$

Sojourn Description

1. Embedded Markov chain of state sequences ξ_i has one step transition probabilities $p_{ij}=\frac{q_{ij}}{q_i}$
2. Successive sojourn times S_{ξ_i} are exponentially distributed with mean $\frac{1}{q_{\xi_i}}$.

Two State Markov Chain

$$
\begin{aligned}
\boldsymbol{A} &= \left\| \begin{array}{cc} -\alpha & \alpha \\ \beta & -\beta \end{array} \right\| \\
\boldsymbol{A}^2 &= \left\| \begin{array}{cc} \alpha^2 + \alpha\beta & -\alpha^2 - \alpha\beta \\ -\beta^2 - \alpha\beta & \beta^2 + \alpha\beta \end{array} \right\| = -(\alpha+\beta) \left\| \begin{array}{cc} -\alpha & \alpha \\ \beta & -\beta \end{array} \right\|
\end{aligned}
$$

Thus

$$\boldsymbol{A}^n = [-(\alpha+\beta)]^{n-1} \boldsymbol{A}$$

And

$$
\begin{aligned}
\boldsymbol{P}(t) &= \boldsymbol{I} - \frac{1}{\alpha+\beta} \sum_{n=1}^{\infty} \frac{[-(\alpha+\beta)t]^n}{n!} \boldsymbol{A} \\
&= \boldsymbol{I} - \frac{1}{\alpha+\beta} \left(e^{-(\alpha+\beta)t} - 1 \right) \boldsymbol{A} \\
&= \boldsymbol{I} + \frac{1}{\alpha+\beta} \boldsymbol{A} - \frac{1}{\alpha+\beta} \boldsymbol{A} e^{-(\alpha+\beta)t}
\end{aligned}
$$

Summary of most Important Results

$$\boldsymbol{P}(t) = e^{\boldsymbol{A}t} \to \mathbf{1}\boldsymbol{\pi} \quad \text{for} \;\; t \to \infty$$

Under an assumption of irreducibility

Additional Reading

Kai Lai Chung: “Markov Chains with Stationary Transition Probabilities”

Queueing process

1. Input Process
2. Service Process
3. Queue Discipline

Kendall notation: $A/B/s$

1. Number of customers/items in system
2. Throughput
3. Utilization
4. Customer waiting time, probability of being served

Little's Theorem

$$L = \lambda W$$

M/M/1 System

Steady state equations:

$$\pi_k = \lim_{t\to\infty} \mathbb{P}\{X(t) = k\} \text{ for } k = 0, 1, \ldots$$

$$\pi_k \mu = \pi_{k-1}\lambda, \qquad \pi_k = \theta_k \pi_0$$

with

$$\theta_k = \left(\frac{\lambda}{\mu}\right)^k$$

such that

$$\pi_0 \sum_{k=0}^{\infty} \theta_k = 1 \Leftrightarrow \pi_0 = \frac{1}{\sum_{k=0}^{\infty} \theta_k} = \frac{1}{\frac{1}{1-\frac{\lambda}{\mu}}} = 1 - \frac{\lambda}{\mu} \text{ for } \lambda < \mu$$

$$\pi_k = \left(\frac{\lambda}{\mu}\right)^k \left(1 - \frac{\lambda}{\mu}\right) = \rho^k(1-\rho) \text{ with } \rho = \frac{\lambda}{\mu}$$

A geometric distribution

Total Time in M/M/1 System

$$\mathbb{P}\{T \le t | n \text{ ahead}\} = \int_0^t \mu \frac{(\mu\tau)^n}{n!} e^{-\mu\tau} d\tau$$

$$\begin{aligned}
\mathbb{P}\{T \le t\} &= \sum_{n=0}^{\infty} \mathbb{P}\{T \le t | n \text{ ahead}\} \mathbb{P}\{n \text{ ahead}\} \\
&= \sum_{n=0}^{\infty} \left[\int_0^t \mu \frac{(\mu\tau)^n}{n!} e^{-\mu\tau} d\tau \right] \left(1 - \frac{\lambda}{\mu}\right) \left(\frac{\lambda}{\mu}\right)^n \\
&= \int_0^t \mu \left(1 - \frac{\lambda}{\mu}\right) e^{-\mu\tau} \sum_{n=0}^{\infty} \frac{(\mu\tau)^n}{n!} \left(\frac{\lambda}{\mu}\right)^n \\
&= \int_0^t (\mu - \lambda) e^{-(\mu-\lambda)\tau} d\tau \\
&= 1 - e^{-(\mu-\lambda)t}
\end{aligned}$$

Mean performance measures

$$W = \frac{1}{\mu - \lambda}$$

From Littles theorem

$$L = \lambda W = \frac{\lambda}{\mu - \lambda} = \frac{\rho}{1 - \rho}$$

which we could also have gotten directly from the distribution π_k

M/M/1 Busy Period

$$\mathbb{E}[I_1] = \frac{1}{\lambda}$$

$$\lim_{t\to\infty} p_0(t) = \pi_0 = \frac{\mathbb{E}[I_1]}{\mathbb{E}[I_1] + \mathbb{E}[B_1]}$$

$$1 - \frac{\lambda}{\mu} = \frac{\frac{1}{\lambda}}{\frac{1}{\lambda} + \mathbb{E}[B_1]}$$

$$\mathbb{E}[B_1] = \frac{1}{\mu - \lambda}$$

M/M/∞ System

$$\lambda_k = \lambda \qquad \mu_k = k\mu$$

$$\pi_k \lambda = \pi_{k+1}(k+1)\mu \qquad \theta_k = \frac{\left(\frac{\lambda}{\mu}\right)^k}{k!}$$

$$\sum_{k=0}^{\infty} \pi_k = \pi_0 \sum_{k=0}^{\infty} \theta_k = \pi_0 \sum_{k=0}^{\infty} \frac{\left(\frac{\lambda}{\mu}\right)^k}{k!} = \pi_0 e^{\frac{\lambda}{\mu}}$$

$$\pi_k = \frac{\left(\frac{\lambda}{\mu}\right)^k}{k!} e^{-\frac{\lambda}{\mu}}$$

a Poisson distribution

M/M/s System

$$\lambda_k = \lambda, \mu_k = \begin{cases} k\mu & \text{for } k < s \\ s\mu & \text{for } s \leq k \end{cases}$$

$$\pi_0 = \left\{ \sum_{j=0}^{s-1} \frac{1}{j!} \left(\frac{\lambda}{\mu} \right)^j + \frac{\left(\frac{\lambda}{\mu} \right)^s}{s! \left(1 - \frac{\lambda}{s\mu} \right)} \right\}$$

$$\pi_k = \begin{cases} \frac{1}{k!} \left(\frac{\lambda}{\mu} \right)^k \pi_0 & \text{for } k = 0, 1, \ldots, s \\ \frac{1}{s!} \left(\frac{\lambda}{\mu} \right)^s \left(\frac{\lambda}{s\mu} \right)^{k-s} \pi_0 & \text{for } k \geq s \end{cases}$$

M/M/s System: Performance Measures

$$
\begin{aligned}
L_0 &= \sum_{j=s}^{\infty}(j-s)\pi_j \\
&= \pi_0 \sum_{k=0}^{\infty} k \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!}\left(\frac{\lambda}{s\mu}\right)^k \\
&= \frac{\pi_0}{s!}\left(\frac{\lambda}{\mu}\right)^s \sum_{k=0}^{\infty} k\left(\frac{\lambda}{s\mu}\right)^k \\
&= \frac{\pi_0}{s!}\left(\frac{\lambda}{\mu}\right)^s \frac{\left(\frac{\lambda}{s\mu}\right)}{\left(1-\frac{\lambda}{s\mu}\right)^2} \\
W_0 &= \frac{L_0}{\lambda} \qquad W = W_0 + \frac{1}{\mu} \qquad L = \lambda W = L_0 + \frac{\lambda}{\mu}
\end{aligned}
$$

M/G/1 System

Let $Y_1, Y_2, \ldots$ be a sequence of service times with cumulative distribution function $G(y)$, with mean $\nu = \mathbb{E}[Y_k]$

$$\lim_{t\to\infty} p_0(t) = \pi_0 = \frac{\mathbb{E}[I_1]}{\mathbb{E}[I_1] + \mathbb{E}[B_1]}$$

$\mathbb{E}[I_1] = \frac{1}{\lambda}$, A: number of arrivals during service time of first customer of the busy period

$$\mathbb{E}[B_1|Y_1 = y, A = 0] = y, \qquad \mathbb{E}[B_1|Y_1 = y, A = 1] = y + \mathbb{E}[B_1]$$

$$\mathbb{E}[B_1|Y_1 = y, A = 1] = y + n\mathbb{E}[B_1]$$

$$\mathbb{E}[B_1|Y_1 = y] = \sum_{n=0}^{\infty}(y + n\mathbb{E}[B_1])\frac{(\lambda y)^n}{n!}e^{-\lambda y} = y + \lambda y\mathbb{E}[B_1]$$

Finally $\mathbb{E}[B_1] = \mathbb{E}[\mathbb{E}[B_1|Y]] = \nu + \nu\lambda\mathbb{E}[B_1]$ such that

$$\mathbb{E}[B_1] = \frac{\nu}{1-\nu\lambda} \text{ and } \pi_0 = \frac{\frac{1}{\lambda}}{\frac{1}{\lambda} + \frac{\nu}{1-\nu\lambda}} = 1 - \nu\lambda \text{ for } \lambda\nu < 1$$

M/G/1 Embedded Markov chain at departures

$$
\begin{aligned}
X_{n+1} &= \begin{cases} X_n - 1 + A_n & \text{for } X_n \geq 1 \\ A_n & \text{for } X_n = 0 \end{cases} \\
&= (X_n - 1)^+ A_n \\
\alpha_k &= \mathbb{P}\{A_n = k\} = \int_0^\infty \frac{(\lambda y)^k}{k!} e^{-\lambda y} \mathrm{d}G(y) \\
P_{ij} &= \begin{cases} \alpha_{j-i-1} & \text{for } j \geq i - 1 \geq 0 \\ \alpha_j & \text{for } i = 0 \end{cases}
\end{aligned}
$$

Mean Queue Length in Equilibrium

$$\lim_{n\to\infty} \mathbb{P}\{X_n = k\} = \pi_k, \qquad \lim_{n\to\infty} \mathbb{E}[X_n] = L$$

$$X_{n+1} = X_n - \delta + A_n, \qquad \mathbb{E}[X_{n+1}] = \mathbb{E}[X_n] - \mathbb{E}[\delta] + \mathbb{E}[A_n]$$

Leading to $\mathbb{E}[A_n] = \mathbb{E}[\delta] = 1 - \pi_0 = \lambda\nu$

Now by Squaring

$$\begin{aligned}
X_{n+1}^2 &= X_n^2 + \delta^2 + A_n^2 - 2X_n\delta + 2A_n(X_n - \delta) \\
\mathbb{E}\left[X_{n+1}^2\right] &= \mathbb{E}\left[X_n^2\right] + \mathbb{E}[\delta] + \mathbb{E}\left[A_n^2\right] - 2\mathbb{E}[X_n] + 2\mathbb{E}[A_n]\mathbb{E}[X_n - \delta] \\
0 &= \lambda\nu + \mathbb{E}\left[A_n^2\right] - 2L + 2\lambda\nu(L - \lambda\nu) \\
L &= \frac{\lambda\nu + \mathbb{E}\left[A_n^2\right] - 2(\lambda\nu)^2}{2(1-\lambda\nu)}
\end{aligned}$$

M/G/1 Calculation of $\mathbb{E}\left[A_n^2\right]$

$$\mathbb{E}\left[A_n^2 | Y = y\right] = (\lambda y + (\lambda y)^2)$$

$$\mathbb{E}\left[A_n^2\right] = \int_0^\infty \left(\lambda y + (\lambda y)^2\right) \mathrm{d}G(y) = \lambda\nu + \lambda^2(\nu^2 + \tau^2)$$

with $\tau^2 = \mathbb{V}\mathrm{ar}(Y)$. Finally, inserting and rearranging

$$L = \frac{\lambda\nu + \lambda\nu + \lambda^2(\nu^2+\tau^2) - 2(\lambda\nu)^2}{2(1-\lambda\nu)} = \rho + \frac{\lambda^2\tau^2 + \rho^2}{2(1-\lambda\nu)}$$

The distributions of $X(t)$ and X_n are identical

M/G/∞ system

$$A_t = \{(w, v) : 0 \leq w \leq t \text{ and } v > t - w\}$$

$$\begin{aligned}
\mu(A_t) &= \int_0^t \left\{ \int_{t-w}^{\infty} \mathrm{d}G(y) \right\} \mathrm{d}w \\
&= \lambda \int_0^t [1 - G(t - w)] \mathrm{d}w \\
&= \lambda \int_0^t [1 - G(x)] \mathrm{d}x
\end{aligned}$$

$$\mathbb{P}\{X(t) = k\} = \frac{(\lambda\mu(A_t))^k}{k!} e^{-\lambda\mu(A_t)} \rightarrow \frac{(\lambda\nu)^k}{k!} e^{-\lambda\nu} \text{ as } t \rightarrow \infty$$