Birth and Death Processes

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Birth and Death Processes

Today:

- Birth processes
- Death processes
- Biarth and death processes
- Limiting behaviour of birth and death processes

Next week

- Finite state continuous time Markov chains
- Queueing theory

Two weeks from now

- Renewal phenomena

Birth and Death Processes

- Birth Processes: Poisson process with intensities that depend on $X(t)$
- Death Processes: Poisson process with intensities that depend on $X(t)$ counting deaths rather than births
- Birth and Death Processes: Combining the two, on the way to continuous time Markov chains/processes

Poisson postulates

i $\mathbb{P}\{X(t+h) - X(t) = 1 | X(t) = x\} = \lambda h + o(h)$

ii $\mathbb{P}\{X(t+h) - X(t) = 0 | X(t) = x\} = 1 - \lambda h + o(h)$

iii $X(0) = 0$

Where

$$\lim_{h \to 0+} \frac{\mathbb{P}\{X(t+h) - X(t) = 1 | X(t) = x\}}{h} = \lambda + \epsilon(h)$$

Birth Process Postulates

i $\mathbb{P}\{X(t+h) - X(t) = 1 | X(t) = k\} = \lambda_k h + o(h)$

ii $\mathbb{P}\{X(t+h) - X(t) = 0 | X(t) = k\} = 1 - \lambda_k h + o(h)$

iii $X(0) = 0$ (not essential, typically used for convenience)

We define

$$P_n(t) = \mathbb{P}\{X(t) = n | X(0) = 0\}$$

Birth Process Differential Equations

$$
\begin{aligned}
P_n(t+h) &= P_{n-1}(t)\left(\lambda_{n-1}h + o(h)\right) + P_n(t)\left(1 - \lambda_n h + o(h)\right) \\
P_n(t+h) - P(t) &= P_{n-1}(t)\lambda_{n-1}h + P_n(t)\lambda_n h + o(h)
\end{aligned}
$$

$$
\begin{aligned}
P_0'(t) &= -\lambda P_0(t) \\
P_n'(t) &= -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t) \text{ for } n \geq 1 \\
P_0(0) &= 1
\end{aligned}
$$

Sojourn times

Define S_k as the time between the kth and $(k+1)$st birth

$$P_n(t) = \mathbb{P}\left\{\sum_{k=0}^{n-1} S_k \leq t < \sum_{k=0}^{n} S_k\right\}$$

where $S_i \sim \exp(\lambda_i)$.
With $W_k = \sum_{i=0}^{k-1} S_i$

$$P_n(t) = \mathbb{P}\{W_n \leq t < W_{n+1}\}$$

$$\mathbb{P}\{S_0 \leq t\} = \mathbb{P}\{W_1 \leq t\} = 1 - \mathbb{P}\{X(t) = 0\} = 1 - P_0(t) = 1 - e^{-\lambda_0 t}$$

Solution of differential equations

Introduce $Q_n(t) = e^{\lambda_n t} P_n(t)$, then

$$\begin{aligned} Q_n'(t) &= \lambda_n e^{\lambda_n t} P_n(t) + e^{\lambda_n t} P_n'(t) \\ &= e^{\lambda_n t} \left(\lambda_n P_n(t) + P_n'(t) \right) \\ &= e^{\lambda_n t} \lambda_{n-1} P_{n-1}(t) \end{aligned}$$

such that

$$Q_n(t) = \lambda_{n-1} \int_0^t e^{\lambda_n x} P_{n-1}(x) \mathrm{d}x$$

leading to

$$P_n(t) = \lambda_{n-1} e^{-\lambda_n t} \int_0^t e^{\lambda_n x} P_{n-1}(x) \mathrm{d}x$$

Regular Process

$$\sum_{n=0}^{\infty} P_n(t) \stackrel{?}{=} 1$$

True if:

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{\lambda_k} = \infty$$

Then

$$\sum_{k=0}^{\infty} P_k(t) = 1$$

Recursive full solution when $\lambda_i \neq \lambda_j$ for $i \neq j$

$$P_n(t) = \left(\prod_{j=0}^{n-1} \lambda_j\right) \sum_{j=0}^{n} B_{j,n} e^{-\lambda_j t}$$

with

$$B_{i,n} = \prod_{j \neq i} \left(\lambda_j - \lambda_i\right)^{-1}$$

Yule Process

$$\begin{aligned} P_n'(t) &= -\beta n P_n(t) + \beta(n-1) P_{n-1}(t) \\ P_n(t) &= e^{-\beta t}\left(1 - e^{-\beta t}\right)^{n-1} \end{aligned}$$

Death Process Postulates

- **i** $\mathbb{P}\{X(t+h) = k-1 | X(t) = k\} = \mu_k h + o(h)$
- **ii** $\mathbb{P}\{X(t+h) = k | X(t) = k\} = 1 - \mu_k h + o(h)$
- **iii** $X(0) = N$

$$P_n(t) = \left(\prod_{j=0}^{n-1} \mu_j\right) \sum_{j=n}^{N} A_{j,n} e^{-\lambda_j t}$$

with

$$A_{k,n} = \prod_{j=n, j\neq k}^{N} \left(\mu_j - \mu_k\right)^{-1}$$

For $\mu_k = k\mu$ we have by a simple probabilistic argument

$$P_n(t) = \binom{N}{n} \left(e^{-\mu t}\right)^n \left(1 - e^{-\mu t}\right)^{N-n} = \binom{N}{n} e^{-n\mu t} \left(1 - e^{-\mu t}\right)^{N-n}$$

Birth and Death Process Postulates

$$P_{ij}(t) = \mathbb{P}\{X(t+s) = j | X(s) = i\} \text{ for all } s \geq 0$$

1. $P_{i,i+1}(h) = \lambda_i h + o(h)$
2. $P_{i,i-1}(h) = \mu_i h + o(h)$
3. $P_{i,i}(h) = -(\lambda_i + \mu_i)h + o(h)$
4. $P_{i,j}(0) = \delta_{ij}$
5. $\mu_0 = 0, \lambda_0 > 0, \mu, \lambda_i > 0, i = 1, 2, \ldots$

Infinitesimal Generator

$$
\boldsymbol{A} = \left\| \begin{array}{ccccc} -\lambda_0 & \lambda_0 & 0 & 0 \ldots & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 \ldots & \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 \ldots & \\ 0 & 0 & \mu_3 & -(\lambda_3+\mu_3) & \ldots \\ \vdots & \vdots & \vdots & \vdots & \end{array} \right\|
$$

$$
P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s), \qquad \boldsymbol{P}(t+s) = \boldsymbol{P}(t)\boldsymbol{P}(s)
$$

Regular Process

$$
\sum_{n=0}^{\infty} \frac{1}{\lambda_n \theta_n} \sum_{k=0}^{n} \theta_k = \infty
$$

where

$$
\theta_0 = 1, \qquad \theta_n = \prod_{k=0}^{n-1} \frac{\lambda_k}{\mu_{k+1}}
$$

Backward Kolomogorov equations

$$
\begin{aligned}
P_{ij}(t+h) &= \sum_{k=0}^{\infty} P_{ik}(h)P_{kj}(t) \\
&= P_{i,i-1}(h)P_{i-1,j}(t) + P_{i,i}(h)P_{i,j}(t) + P_{i,i+1}(h)P_{i+1,j}(t) + o(h) \\
&= \mu_i h P_{i-1,j}(t) + (1-(\mu_i+\lambda_i)h)P_{i,j}(t) + \lambda_i h P_{i+1,j}(t) + o(h)
\end{aligned}
$$

ODE's for Birth and Death Process

$$
\begin{aligned}
P'_{0j}(t) &= -\lambda_0 P_{0j}(t) + \lambda_1 P_{1j}(t) \\
P'_{ij}(t) &= \mu_i P_{i-1,j}(t) - (\lambda_i + \mu_i) P_{ij}(t) + \lambda_i P_{i+1,j}(t) \\
P_{ij}(0) &= \delta_{ij} \\
\boldsymbol{P}'(t) &= \boldsymbol{A}\boldsymbol{P}(t)
\end{aligned}
$$

Forward Kolmogorov equations

$$P_{ij}(t+h) = \sum_{k=0}^{\infty} P_{ik}(t)P_{kj}(h)$$

$$\boldsymbol{P}'(t) = \boldsymbol{P}(t)\boldsymbol{A}$$

The backward and forward equations have the same solutions in all “ordinary” models, that is models without explosion and models without instantenuous states

ODE's for Birth and Death Process

$$
\begin{aligned}
P'_{i0}(t) &= -P_{i0}(t)\lambda_0 + P_{i1}(t)\mu_1 \\
P'_{ij}(t) &= P_{i,j-1}\lambda_{j-1} - P_{jj}(t)(\lambda_j + \mu_j) + P_{i,j+1}(t)\mu_{j+1} \\
P_{ij}(0) &= \delta_{ij} \\
\boldsymbol{P}'(t) &= \boldsymbol{AP}
\end{aligned}
$$

Sojourn times

$$\mathbb{P}\{S_i \geq t\} = G_i(t)$$

$$\begin{aligned} G_i(t+h) &= G_i(t)G_i(h) = G_i(t)[P_{ii}(h) + o(h)] \\ &= G_i(t)[1 - (\lambda_i + \mu_i)h] + o(h) \end{aligned}$$

$$G_i'(t) = -(\lambda_i + \mu_i)G_i(t)$$

$$G_i(t) = e^{-(\lambda_i + \mu_i)t}$$

Embedded Markov chain

Define T_n as the time of the nth state change at the Define $N(t)$ to be number of state changes up to time t.

$$\mathbb{P}\{X(T_{n+1}) = j | X(T_n) = i\}$$

Define $Y_n = X(T_n)$

$$\mathbb{P}\{Y_{n+1} = j | Y_n = i\} = \begin{cases} \frac{\mu_i}{\mu_i+\lambda_i} & \text{for } j = i-1 \\ \frac{\lambda_i}{\mu_i+\lambda_i} & \text{for } j = i+1 \\ 0 & \text{for } j \notin \{i-1, i+1\} \end{cases}$$

$$\boldsymbol{P} = \left\| \begin{array}{ccccc} 0 & 1 & 0 & 0 & \dots \\ \frac{\mu_1}{\mu_1+\lambda_1} & 0 & \frac{\lambda_1}{\mu_1+\lambda_1} & 0 & \dots \\ 0 & \frac{\mu_2}{\mu_2+\lambda_2} & 0 & \frac{\lambda_2}{\mu_2+\lambda_2} & \dots \\ 0 & 0 & \frac{\mu_3}{\mu_3+\lambda_3} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right\|$$

Definition through Sojourn Times and Embedded Markov Chain

Sequence of states governed by the discrete Time Markov chain with transition probability matrix $\boldsymbol{P}$

Exponential sojourn times in each state with intensityparameter $\gamma_i (= \mu_1 + \lambda_i)$

Linear Growth with Immigration

$$
\begin{aligned}
P'_{i0}(t) &= -aP_{i0}(t) + \mu P_{i1}(t) \\
P'_{ij}(t) &= [\lambda(j-1)+a]P_{i,j-1}(t) - [(\lambda+\mu)j+a]P_{ij}(t) + \mu(j+1)P_{i,j+1}(t)
\end{aligned}
$$

With $M(0) = i$ if $X(0)$ this leads to

$$
\begin{aligned}
\mathbb{E}[X(t)] &= M(t) = \sum_{j=1}^{\infty} jP_{ij}(t) \\
M'(t) &= a + (\lambda-\mu)M(t) \\
M(t) &= \begin{cases} at + i & \text{if } \lambda = \mu \\ \frac{a}{\lambda-\mu}\left\{e^{(\lambda-\mu)t} - 1\right\} + ie^{(\lambda-\mu)t} & \text{if } \lambda \neq \mu \end{cases}
\end{aligned}
$$

Two-State Markov Chain

$$\boldsymbol{A} = \left\| \begin{array}{cc} -\alpha & \alpha \\ \beta & -\beta \end{array} \right\|$$

$$P'_{00}(t) = -\alpha P_{00}(t) + \beta P_{01}(t)$$

With $P_{01}(t) = 1 - P_{00}(t)$ we get

$$P'_{00}(t) = -(\alpha + \beta)P_{00}(t) + \beta$$

Using the standard approach with $Q_{00}(t) = e^{(\alpha+\beta)t}P_{00}(t)$ we get

$$Q_{00}(t) = \frac{\beta}{\alpha + \beta}e^{(\alpha+\beta)t} + C$$

which with $P_{00}(0) = 1$ give us

$$P_{00}(t) = \frac{\beta}{\alpha + \beta} + \frac{\alpha}{\alpha + \beta}e^{-(\alpha+\beta)t} = \pi_1 + \pi_2 e^{-(\alpha+\beta)t}$$

with $\boldsymbol{\pi} = (\pi_1, \pi_2) = \left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right)$.

Two-State Markov Chain - continued

Using $P_{01}(t) = 1 - P_{00}(t)$ we get

$$P_{01}(t) = \pi_2 - \pi_2 e^{-(\alpha+\beta)t}$$

and by an identical derivation

$$\begin{aligned} P_{11}(t) &= \pi_2 + \pi_1 e^{-(\alpha+\beta)t} \\ P_{10}(t) &= \pi_1 - \pi_1 e^{-(\alpha+\beta)t} \end{aligned}$$

Limiting Behaviour for Birth and Death Processes

For an irreducible birth and death process we have

$$\lim_{t\to\infty} P_{ij}(t) = \pi_j \geq 0$$

If $\pi_j > 0$ then

$$\boldsymbol{\pi}\boldsymbol{P}(t) = \boldsymbol{\pi} \quad \text{or} \quad \boldsymbol{\pi}\boldsymbol{A} = \boldsymbol{0}$$

We can always solve recursively for $\boldsymbol{\pi}$

$$\pi_n\lambda_n = \pi_{n+1}\mu_{n+1}$$

such that

$$\pi_n = \left(\prod_{i=0}^{n-1}\frac{\lambda_i}{\mu_{i+1}}\right)\pi_0$$

such that

$$\pi_0 = \left[1 + \sum_{n=1}^{\infty}\left(\prod_{i=0}^{n-1}\frac{\lambda_i}{\mu_{i+1}}\right)\right]^{-1}$$

Linear Growth with Immigration

$\lambda_n = n\lambda + a$, $\mu_n = n\mu$ With

$$\begin{aligned}
\theta_k &= \prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}} \\
&= \frac{a(a+\lambda)\cdots(a+(k-1)\lambda}{k!\mu^k} \\
&= \frac{\frac{a}{\lambda}(\frac{a}{\lambda}+1)\cdots(\frac{a}{\lambda}+(k-1)}{k!}\left(\frac{\lambda}{\mu}\right)^k \\
&= \binom{\frac{a}{\lambda}+k-1}{k}\left(\frac{\lambda}{\mu}\right)^k
\end{aligned}$$

$$\begin{aligned}
\sum_{k=0}^{\infty}\theta_k &= \sum_{k=0}^{\infty}\binom{\frac{a}{\lambda}+k-1}{k}\left(\frac{\lambda}{\mu}\right)^k = \left(1-\frac{\lambda}{\mu}\right)^{\frac{a}{\lambda}} \\
\pi_k &= \binom{\frac{a}{\lambda}+k-1}{k}\left(\frac{\lambda}{\mu}\right)^k\left(1-\frac{\lambda}{\mu}\right)^{\frac{a}{\lambda}}
\end{aligned}$$

Logistic Model

Birth/death rate per individual

- $\lambda = \alpha(M - X(t))$
- $\mu = \beta(X(t) - N)$,

such that $\lambda_n = \alpha n(M-n), \mu_n = \beta n(n-N)$.

$$\theta_{N+m} = \left(\frac{\alpha}{\beta}\right)^m \prod_{i=N}^{N+m-1} \frac{i(M-i)}{(i+1)(i+1-N)}$$

$$= \frac{N}{N+m}\binom{M-N}{m}\left(\frac{\alpha}{\beta}\right)^m, 0 \le m \le M-N$$

$$\pi_{N+M} = \frac{c}{N+m}\binom{M-N}{m}\left(\frac{\alpha}{\beta}\right)^m$$

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