Poisson Processes

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02407 Stochastic Processes 4, 25 September 2018

Poisson Process

Today:

- Definition (in terms of Poisson distribution)
- The law of rare events
- Definition in terms of intervals
- Uniformity
- Spatial, compound, and marked Poisson processes

Next week

- Birth processes
- Death processes
- Birth and death processes

Two weeks from now

- Limiting behaviour of birth and death processes
- Birth and death processes with absorption
- Finite continuous time Markov chains

The Poisson process

- The next fundamental process
- As important as discrete time Markov chain
- Continuous time model
- Parallel to Bernoulli process
- Model for complete randomness
- Three different characterisations

The Bernoulli Process

Definition
A Bernoulli process of parameter, or rate, p, is an integer-indexed integer-valued stochastic process $\{X(t); t \geq 0\}$ *for which*

1. *For any time points* $t_0 = 0 < t_1 < t_2 < \cdots < t_n$*, the process increments*

$$X(t_1) - X(t_0), X(t_2) - X(t_1), \ldots X(t_n) - X(t_{n-1})$$

   are independent random variables;

2. *For* $s \geq 0$ *and* $t \geq 0$*, the random variable* $X(t+s) - X(s)$ *has the binomial distribution*

$$\mathbb{P}\{X(t+s) - X(s) = k\} = \binom{t}{k} p^k (1-p)^{t-k}$$

3. $X(0) = 0$.

Bernoulli Process Waiting times and intensities

Define W_n waiting time to the nth event

1. The waiting time (W_1) to the first event (and the waiting time $W_{n+1} - W_n$ between the n and the $(n+1)$st event) is geometric, $\mathbb{P}\{W_1 = k\} = p(1-p)^{k-1}, \mathbb{P}\{W_1 > k\} = (1-p)^k, k = 1, 2, 3, \ldots$.
2. The waiting time to the nth event follows a negative binomial distribution

$$\mathbb{P}\{W_n = k\} = \binom{k-1}{n-1} p^n (1-p)^{k-n}, \text{ for } k = n, n+1, \ldots$$

The Probability (intensity) of having an event in a single interval is

1. $\mathbb{P}\{X(t+1) - X(t) = 0\} = 1 - p$
2. $\mathbb{P}\{X(t+1) - X(t) = 1\} = p$
3. $\mathbb{P}\{X(t+1) - X(t) > 1\} = 0$

Poisson process - three different characterisations

- Independence assumptions in all cases
  - Number of events in an interval is Poisson distributed with independent increments
  - Constant intensity for event
  - Intervals between event are exponentially distributed

The Poisson Process

Definition (Page 225)

A Poisson process of intensity, or rate, $\lambda > 0$, is an integer-valued stochastic process $\{X(t); t \geq 0\}$ for which

1. *For any time points $t_0 = 0 < t_1 < t_2 < \cdots < t_n$, the process increments*

$$X(t_1) - X(t_0), X(t_2) - X(t_1), \ldots X(t_n) - X(t_{n-1})$$

are independent random variables;

2. *For $s \geq 0$ and $t \geq 0$, the random variable $X(t+s) - X(s)$ has the Poisson distribution*

$$\mathbb{P}\{X(t+s) - X(s) = k\} = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \text{ for } k = 0, 1, \ldots$$

3. $X(0) = 0$.

Intensity characterisation - infinitessimal probabilities

- $\mathbb{P}\{\text{one event in an interval of length } \Delta t\}$
  $\mathbb{P}\{X(t+\Delta t) - X(t) = 1\} = \Delta t \lambda + \text{some small quantity}$
  $= \lambda \Delta t + o(\Delta t)$ Where

$$\frac{o(\Delta t)}{\Delta t} \to 0 \text{ as } \Delta t \to 0$$

  $o(t)$ is a function that tends to zero faster than t
- $\mathbb{P}\{\text{no event in an interval of length } \Delta t\}$
  $\mathbb{P}\{X(t+\Delta t) - X(t) = 0\} = 1 - \Delta t \lambda + o(\Delta t)$
- $\mathbb{P}\{\text{more than one event during } \Delta t\} = o(\Delta t)$

Nonhomogeneous Poisson Process

$$\begin{aligned}\mathbb{P}\{X(t+h)-X(t)=1\} &= \frac{(\lambda h)}{1!}e^{-\lambda h} \\ &= (\lambda h)\left(1-\lambda h+\frac{1}{2}\lambda^2h^2-\cdots\right) \\ &= \lambda h+o(h)\end{aligned}$$

where $\lim_{h\to 0}\frac{o(h)}{h}=0$ If we assume

$$\begin{aligned}\mathbb{P}\{X(t+h)-X(t)=1\} &= \lambda_t(h)+o(h) \\ \mathbb{P}\{X(t+h)-X(t)=0\} &= 1-\lambda_t(h)+o(h)\end{aligned}$$

$$X(t)-X(s)\sim \text{Pois}\left(\int_s^t \lambda(u)\mathrm{d}u\right)$$

Homogeneity transformation $Y(s)=X(t)$ with $s=\int_0^t\lambda(u)\mathrm{d}u$

The Law of Rare Events

Theorem (*5.3 Page 233)*

Let $\epsilon_1,\epsilon_2,\ldots$ be independent Bernoulli variables, where

$$\mathbb{P}\{\epsilon_i=1\}=p_i \text{ and } \mathbb{P}\{\epsilon_i=1\}=1-p_i$$

and let $S_n=\epsilon_1+\cdots+\epsilon_n$. The exact probabilities for S_n are given by

$$\mathbb{P}\{S_n=k\}=\sum^{(k)}\prod_{i=1}^n p_i^{x_i}(1-p_i)^{1-x_i},$$

where $\sum^{(k)}$ denotes the sum over all 0,1 valued x_i's such that $x_1+\cdots+x_n=k$, and Poisson probabilities with $\mu=p_1+\cdots+p_n$ differ at most by

$$\left|\mathbb{P}\{S_n=k\}-\frac{\mu^k}{k!}e^{-\mu}\right|\le\sum_{i=1}^n p_i^2$$

□

From Poisson distribution to exponential distribution

- $X(t)\in P(\lambda t)$ $\quad \mathbb{P}\{X(t)=n\}=\frac{(\lambda t)^n}{n!}e^{-\lambda t}$
- $\mathbb{P}\{S_0>t\}=\mathbb{P}\{X(t)=0\}=e^{-\lambda t}$
- Such that $P\{S_0\le t\}=F(t)=1-\mathbb{P}\{S_0>t\}=1-e^{-\lambda t}$ the exponential distribution

Exponentially distributed intervals

- W_n time of the nth event
- S_n time between the nth and $n+1$st event (sojourn time).
- The time between two consecutive events is exponentially distributed

$$S_n\in\exp(\lambda) \qquad \mathbb{P}\{S_n\le t\}=F(t)=1-e^{-\lambda t}$$

- The intervals are iid.
- Important relation between S_n and $X(t)$
  - $$\mathbb{P}\{W_n\le t\}=\mathbb{P}\{X(t)\ge n\}$$

Time to the nth event - the Erlang distribution

- The relation between W_n and $X(t)$

$$\mathbb{P}\{W_n \leq t\} = \mathbb{P}\{X(t) \geq n\}$$

$$\mathbb{P}\{W_n \leq t\} = \mathbb{P}\{X(t) \geq n\} = \sum_{i=n}^{\infty} \frac{(\lambda t)^i}{i!} e^{-\lambda t} = 1 - \sum_{i=0}^{n-1} \frac{(\lambda t)^i}{i!} e^{-\lambda t}$$

- $\mathbb{E}(S_n) = \frac{1}{\lambda} \qquad \mathbb{E}(W_n) = \frac{n}{\lambda}$
- $\mathbb{V}\text{ar}(S_n) = \frac{1}{\lambda^2} \qquad \mathbb{V}\text{ar}(W_n) = \frac{n}{\lambda^2}$
- We say that $W_n \in Erl_n(\lambda)$
- The Erlang distribution can be interpreted as the distribution for the sum of independent exponential random variables.

Uniform Distribution and the Poisson Process

Theorem (*5.7 Page 248)*

Let $W_1, W_2, \ldots$ be the occurrence times in a Poisson process $X(t)$ of rate $\lambda > 0$. Conditioned on $X(t) = n$ the random variables $W_1, W_2, \ldots$ have the joint probability density function

$$f_{W_1,\ldots,W_n|X(t)=n}(w_1, \ldots, w_n) = n!t^n \text{ for } 0 < w_1 < \cdots < w_n < t$$

□

$$M = \mathbb{E}\left[\sum_{k=1}^{X(t)} e^{-\beta W_k}\right]$$

$$M = \sum_{n=1}^{\infty} \mathbb{E}\left[\sum_{k=1}^{n} e^{-\beta W_k} | X(t) = n\right] Pr\{X(t) = n\}$$

$$\mathbb{E}\left[\sum_{k=1}^{X(t)} e^{-\beta W_k} | X(t) = n\right] = \mathbb{E}\left[\sum_{k=1}^{n} e^{-\beta U_k}\right]$$

Shot Noise

$$I(t) = \sum_{k=1}^{X(t)} h(t - W_k)$$

$$\mathbb{E}(I(t)) = \lambda \int_0^t h(u)\text{d}u, \; \mathbb{V}\text{ar}(I(t)) = \lambda \int_0^t h(u)^2\text{d}u$$

Spatial Poisson Processes

1. For each A in $\mathcal{A}$, the random variable $N(A)$ has a Poisson distribution with parameter $\lambda|A|$
2. For every finite collection $\{A_1,\ldots,A_n\}$ of disjoint subsets of S, the random variables $N(A_1),\ldots,N(A_n)$ are independent

1. The possible values for $N(A)$ are the nonnegative integers $\{0,1,2,\ldots\}$ and $0 < Pr\{N(A)=0\} < 1$ if $0 < |A| < \infty$
2. The probability distribution of $N(A)$ depends on the set A only through its size (length, area, volume) $|A|$, with the further property that $Pr\{N(A)\geq 1\} = \lambda|A| + o(|A|)$ as $|A| \to 0$.
3. For $m = 2,3,\ldots$, if $A_1, A_2, \ldots, A_m$ are disjoint regions, then $\mathbb{N}(A_1), N(A_2), \ldots, N(A_m)$ are independent random variables and $N(A_1 \cup \mathbb{A}_2 \cup \cdots \cup A_m) = N(A_1) + N(A_2) + \cdots + N(A_m)$.
4. $\lim_{|A|\to 0} \frac{Pr\{N(A)\geq 1\}}{Pr\{N(A)=1\}} = 1$.

Conditional uniform distribution

$$Pr\{N(B)=1|N(A)=1\} = \frac{|B|}{|A|} \text{ for any set } B \subset A$$

For $A_1 \cup A_2 \cup \cdots A_m = A$

$$Pr\{N(A_1)=k_1,\ldots,N(A_m)=k_m|N(A)=n\} = \frac{n!}{k_1!\cdots k_m!}\left(\frac{|A_1|}{|A|}\right)^{k_1}\cdots\left(\frac{|A_m|}{|A|}\right)^{k_m}$$

Compound (Reward) Poisson Processes

We have random variables $Y_1, Y_2, \ldots$ with cumulative distribution function

$$G(y) = \mathbb{P}\{Y_k \leq y\}, \qquad \mathbb{E}(Y_i) = \mu, \qquad \mathbb{V}\mathrm{ar}(Y_i) = \nu^2$$

A Compound Poisson Process (reward process) is defined by

$$Z(t) = \sum_{k=1}^{X(t)} Y_k$$

$\mathbb{E}[Z(t)] = \mu\lambda t, \mathbb{V}\mathrm{ar}[Z(t)] = \lambda t(\mu^2+\nu^2)$

$$\mathbb{P}\{Y_1+\cdots+Y_n \leq y\} = \int_{-\infty}^{\infty} G^{(n-1)}(y-z)\mathrm{d}G(z)$$

$$\begin{aligned}
\mathbb{P}\{Z(t)\leq z\} &= \mathbb{P}\left\{\sum_{k=1}^{X(t)} Y_k \leq z\right\} \\
&= \sum_{n=0}^{\infty} \mathbb{P}\left\{\sum_{k=1}^{X(t)} Y_k \leq z \,\middle|\, X(t)=n\right\}\frac{(\lambda t)^n}{n!}e^{-\lambda t} \\
&= \sum_{n=0}^{\infty} \frac{(\lambda t)^n e^{-\lambda t}}{n!} G^{(n)}(z)
\end{aligned}$$

Shock Model

Let T be the time to get beyond critical level a

$$\{T > t\} \text{ if and only if } \{Z(t) \leq a\}$$

$$\begin{aligned} \mathbb{E}[T] &= \int_0^\infty \mathbb{P}\{T > t\}\mathrm{d}t \\ &= \sum_{n=0}^\infty \left(\int_0^\infty \frac{(\lambda t)^n e^{-\lambda t}}{n!} \mathrm{d}t \right) G^{(n)}(a) \\ &= \lambda^{-1} \sum_{n=0}^\infty G^{(n)}(a) \end{aligned}$$

Marked Poisson Processes

Additional Reading

Erhan Çinlar: “Introduction to Stochastic Processes”