Discrete Time Markov Chains, Limiting Distribution and Classification

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Discrete time Markov chains

Today:

- Discrete time Markov chains - invariant probability distribution
- Classification of states
- Classification of chains

Next week

- Poisson process

Two weeks from now

- Birth- and Death Processes

Regular Transition Probability Matrices

$$\mathbf{P} = \left|\left|P_{ij}\right|\right|, \qquad 0 \leq i, j \leq N$$

Regular: If $\mathbf{P}^k > 0$ for some k

In that case $\lim_{n\to\infty} P_{ij}^{(n)} = \pi_j$

Theorem 4.1 (Page 168) let $\mathbf{P}$ be a regular transition probability matrix on the states $0, 1, \ldots, N$. Then the limiting distribution $\boldsymbol{\pi} = (\pi_0, \pi_1, \pi_N)$ is the unique nonnegative solution of the equations

$$\pi_j = \sum_{k=0}^{N} \pi_k P_{kj}, \qquad \boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$$

$$\sum_{k=0}^{N} \pi_k = 1, \qquad \boldsymbol{\pi}\mathbf{1} = 1$$

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- Long term averages $\lim_{n\to\infty} \frac{1}{1} \sum_{n=1}^{m} P_{ij}^{(n)} = \pi_j$
- Stationary distribution $\boldsymbol{\pi} = \boldsymbol{\pi}\boldsymbol{P}$

A Social Mobility Example

		Son's Class		
		Lower	Middle	Upper
	Lower	0.40	0.50	0.10
Father's	Middle	0.05	0.70	0.25
Class	Upper	0.05	0.50	0.45

$$\boldsymbol{P}^8 = \begin{Vmatrix} 0.0772 & 0.6250 & 0.2978 \\ 0.0769 & 0.6250 & 0.2981 \\ 0.0769 & 0.6250 & 0.2981 \end{Vmatrix}$$

$$\begin{aligned} \pi_0 &= 0.40\pi_0 + 0.05\pi_1 + 0.05\pi_2 \\ \pi_1 &= 0.50\pi_0 + 0.70\pi_1 + 0.50\pi_2 \\ \pi_2 &= 0.10\pi_0 + 0.25\pi_1 + 0.45\pi_2 \\ 1 &= \pi_0 + \pi_1 + \pi_2 \end{aligned}$$

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  - i communicates with j and j communicates with k then i and k communicates

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- (c) i is null persistent (null recurrent) if and only if j is null persistent

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Lemma

If S is finite, then at least one state is persistent(recurrent) and all persistent states are non-null (positive recurrent) □

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$$\lim_{n \to \infty} P_{ii}^{(n)} = \frac{1}{\sum_{n=0}^{\infty} n f_{ii}^{(n)}} = \frac{1}{m_i}$$

(b) under the same conditions as in (a), $\lim_{n \to \infty} P_{ji}^{(n)} = \lim_{n \to \infty} P_{ii}^{(n)}$ for all j.

An example chain (random walk with reflecting barriers)

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$$\boldsymbol{P} = \begin{bmatrix} 0.6 & 0.4 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.3 & 0.3 & 0.4 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.3 & 0.3 & 0.4 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.3 & 0.3 & 0.4 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.3 & 0.3 & 0.4 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.3 & 0.3 & 0.4 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.3 & 0.3 & 0.4 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.3 & 0.7 \end{bmatrix}$$

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With initial probability distribution $\boldsymbol{p}^{(0)} = (1, 0, 0, 0, 0, 0, 0, 0)$ or $X_0 = 1$.

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- Thus all states communicate and the chain is irreducible. Generally we won't bother with bounds for the f_{ij}'s.

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- Thus all states communicate and the chain is irreducible. Generally we won't bother with bounds for the f_{ij}'s.
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Properties of that chain

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- A look on the behaviour of the chain

A number of different sample paths X_n's

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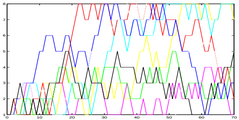

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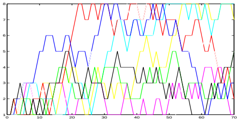

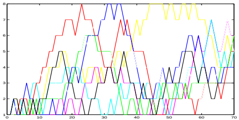

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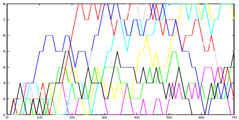

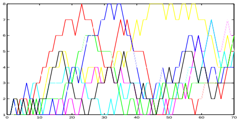

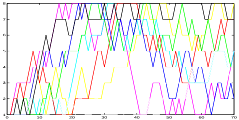

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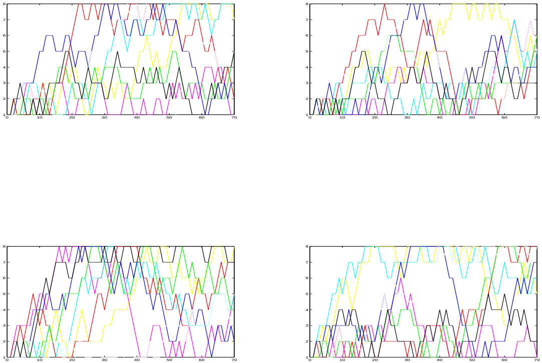

The state probabilities

$p_j^{(n)}$

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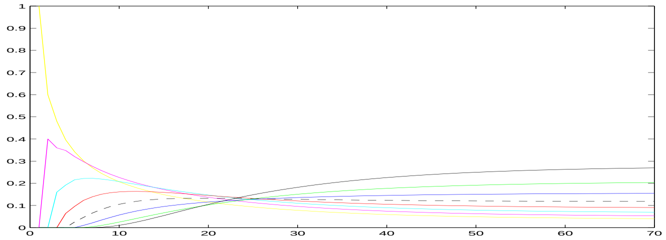

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The example chain (random walk with reflecting barriers)

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$$\boldsymbol{P} = \begin{bmatrix} 0.6 & 0.4 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.3 & 0.3 & 0.4 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.3 & 0.3 & 0.4 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.3 & 0.3 & 0.4 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.3 & 0.3 & 0.4 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.3 & 0.3 & 0.4 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.3 & 0.3 & 0.4 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.3 & 0.7 \end{bmatrix} \qquad \pi = \pi \boldsymbol{P}$$

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$$\pi_1 \quad = \quad \pi_1 \cdot 0.6 +$$

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DTU

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0.0 & 0.3 & 0.3 & 0.4 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.3 & 0.3 & 0.4 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.3 & 0.3 & 0.4 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.3 & 0.3 & 0.4 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.3 & 0.3 & 0.4 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.3 & 0.7
\end{bmatrix}
\qquad \boldsymbol{\pi} = \pi \boldsymbol{P}
$$

Elementwise the matrix equation is $\pi_i = \sum_j \pi_j p_{ji}$

$$
\begin{aligned}
\pi_1 &= \pi_1 \cdot 0.6 + \pi_2 \cdot 0.3 \\
\pi_2 &= \pi_1 \cdot 0.4 + \pi_2 \cdot
\end{aligned}
$$

The example chain (random walk with reflecting barriers)

$$
\boldsymbol{P} = \begin{bmatrix}
0.6 & 0.4 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.3 & 0.3 & 0.4 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.3 & 0.3 & 0.4 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.3 & 0.3 & 0.4 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.3 & 0.3 & 0.4 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.3 & 0.3 & 0.4 & 0.0 \\
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DTU

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DTU

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Or

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$$

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$$\begin{aligned}\pi_2 &= \frac{1-0.6}{0.3}\pi_1 \\ \pi_{j+1} &= \frac{1}{0.3}((1-0.3)\pi_j - 0.4\pi_{j-1})\end{aligned}$$

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$$

Can be solved recursively to find:

$$
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Such that

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$$1 = \pi_1 \frac{1 - \left(\frac{0.4}{0.3}\right)^8}{1 - \frac{0.4}{0.3}} \Leftrightarrow \pi_1 = \frac{1 - \frac{0.4}{0.3}}{1 - \left(\frac{0.4}{0.3}\right)^8}$$

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- Recall the definition of eigenvalues/eigen vectors

DTU

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- Recall the definition of eigenvalues/eigen vectors
- If $\boldsymbol{A}\boldsymbol{y} = \lambda\boldsymbol{y}$ we say that λ is an eigenvalue of $\boldsymbol{A}$ with an associated eigenvector $\boldsymbol{y}$. Here $\boldsymbol{y}$ is a right eigenvector, there is also a left eigenvector

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The solution of $\pi = \pi \boldsymbol{P}$ continued

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- But we have the normalising condition by $\sum_j \pi_j = 1$ this can expressed as $\boldsymbol{\pi}\mathbf{1} = 1$. Where

$$\mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

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For an irreducible Markov chain, (the condition we need to verify)

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- The long run average probability of finding the Markov chain in state i is π_i. $\pi_i = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} p_i^{(k)}$ also true for periodic chains.

Example (null-recurrent) chain

$$\boldsymbol{P} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & \dots \\ 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

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For $p_j > 0$ the chain is obviously irreducible.

Example (null-recurrent) chain

$$\boldsymbol{P} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & \dots \\ 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

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π_1

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π_2

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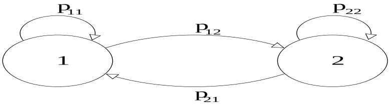
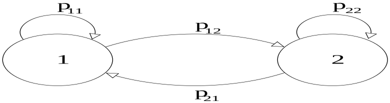

P11
P12
P22
1
2
P21



$$\boldsymbol{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

Reversible Markov chains

- Solve sequence of linear equations instead of the whole system

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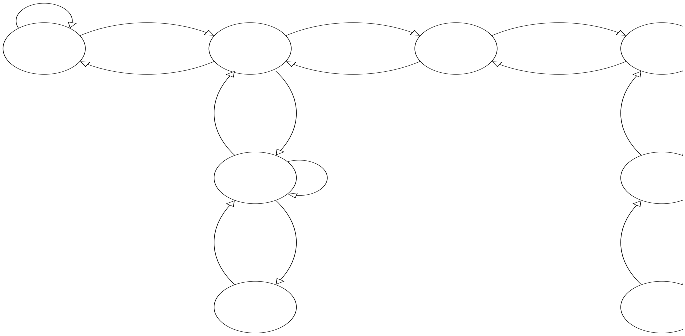

Local balance equations

π_i

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$$\pi_i = \sum_j \pi_j$$

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If they are fulfilled for each i and j, the global balance equations can be obtained by summation.

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For a reversible chain (local balance) this is $\pi_i p_{ij} = \pi_j p_{ji} = \mathbb{P}\{X_{n-1} = j\}\mathbb{P}\{X_n = i | X_{n-1} = j\} = \mathbb{P}\{X_{n-1} = j \cap X_n = i\}$ the reversed sequence.

Another look at a similar question

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$$\mathbb{P}\{X_{n-1} = j | X_n = i\}$$

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$$\mathbb{P}\{X_{n-1} = j | X_n = i\} = \frac{\mathbb{P}\{X_{n-1} = j \cap X_n = i\}}{}$$

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Exercise 10 (16/12/91 ex.1)

In connection with an examination of the reliability of some software intended for use in control of modern ferries one is interested in examining a stochastic model of the use of a control program.
The control program works as " state machine " i.e. it can be in a number of different levels, 4 are considered here. The levels depend on the physical state of the ferry. With every shift of time unit while the program is run, the program will change from level j to level k with probability p_{jk}.

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The program starts running in level 1, and it is known that the program contains one critical error.

DTU

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$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & p_{11} & p_{12} & p_{13} & p_{14} & 0 & 0 & 0 & 0 \\ 0 & p_{21} & p_{22} & p_{23} & p_{24} & 0 & 0 & 0 & 0 \\ 0 & p_{31} & p_{32} & p_{33} & p_{34} & 0 & 0 & 0 & 0 \\ 0 & p_{41} & p_{42} & p_{43} & p_{44} & 0 & 0 & 0 & 0 \end{bmatrix}$$

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	1	Positive recurrent
	2	Positive recurrent
	3	Positive recurrent
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	5	Transient
	6	Transient
	7	Transient
	8	Transient

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$$\sum_{i=1}^{4} 3^{i-1}\pi_1 = 1 \Leftrightarrow 40\pi_1 = 1$$

$$\pi_1 = \frac{1}{40}$$

The sum $\sum_{i=1}^{4} 3^{i-1}$ can be obtained by using $\sum_{i=1}^{4} 3^{i-1} = \frac{1-3^4}{1-3} = 40$.

$$\sum_{i=1}^{4} 3^{i-1}\pi_1 = 1 \Leftrightarrow \frac{1-3^4}{1-3}\pi_1 = 1$$