Variations and Brownian Motion with drift

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Brownian Motion

Today:

- Various variations of Brownian motion, reflected, absorbed, Brownian bridge, with drift, geometric

Next week

- General course overview

Reflected Brownian Motion

$$
\begin{aligned}
R(t) &= \begin{cases} B(t) & \text{if } B(t) \geq 0 \\ -B(t) & \text{if } B(t) < 0 \end{cases} \\
\mathbb{E}(R(t)) &= \sqrt{2t/\pi} \\
\mathbb{V}\text{ar}(R(t)) &= \left(1 - \frac{2}{\pi}\right) t \\
\mathbb{P}\{R(t) \leq y | R(0) = x\} &= \int_{-y}^{y} \phi_t(z - x) \mathrm{d}x \\
p(y, t|x) &= \phi_t(y - x) + \phi_t(y + x)
\end{aligned}
$$

Absorbed Brownian Motion

The movement ceases once the level 0 is reached.

$$
\begin{aligned}
G_t(x, y) &= \mathbb{P}\{A(t) > y | A(0) = x\} \\
&= \mathbb{P}\{B(t) > y, \min\{B(u) > 0; 0 \leq u \leq t | B(0) = x\}
\end{aligned}
$$

We first observe

$$\mathbb{P}\{B(t) > y | B(0) = x\} = G_t(x, y)$$

$$+\mathbb{P}\{B(t) > y, \min\{B(u) \leq 0; 0 \leq u \leq t\} | B(0) = x\}$$

Due to reflection the latter term is also

$$\mathbb{P}\{B(t) > y, \min\{B(u) \leq 0; 0 \leq u \leq t\} | B(0) = x\}$$

$$= \mathbb{P}\{B(t) < -y, \min\{B(u) \leq 0; 0 \leq u \leq t\} | B(0) = x\}$$

$$= \mathbb{P}\{B(t) < -y | B(0) = x\}$$

Absorbed Brownian Motion

Summarizing we get

$$\begin{aligned}\mathbb{P}\{A(t) > y|A(0) = x\} &= G_t(x,y)\\ &= 1 - \Phi_t(y-x) - \Phi_t(-y-x)\\ &= \Phi_t(y+x) - \Phi_t(y-x)\end{aligned}$$

We have already seen that

$$\mathbb{P}\{A(t) = 0|A(0) = x\} = 2[1 - \Phi_t(x)]$$

Brownian Bridge

Distribution of $B(t); 0 \le t \le 1$ conditioned on $\{B(0) = 0, B(1) = 0\}$.
$X = B(t)$ and $B(1) - B(t)$ are indepedent normal variables.
$Y = B(1) - B(t) + B(t)$ and X are bivariate normal.

$$\begin{aligned}\mathbb{V}\mathrm{ar}(x) &= t\\ \mathbb{C}\mathrm{ov}(X,Y) &= \mathbb{E}(B(t)B(t) + B(t)(1-B(t))) = t = \rho\sqrt{t}\sqrt{1}\\ \mathbb{E}(X|Y=y=0) &= \mathbb{E}(X) + \rho\frac{\sigma_X}{\sigma_Y}\cdot(y - \mathbb{E}(Y)) = 0\\ \mathbb{V}\mathrm{ar}(X|Y) &= t(1-\rho^2) = t(1-t)\end{aligned}$$

$$\begin{aligned}\mathbb{E}\{B(t)|B(0)=0, B(1)=0\} &= 0\\ \mathbb{V}\mathrm{ar}\{B(t)|B(0)=0, B(1)=0\} &= t(1-t)\\ \mathbb{C}\mathrm{ov}\{B(s), B(t)|B(0)=0, B(1)=0\} &= s(1-t)\end{aligned}$$

The process is Gaussian

Brownian Motion with Drift

$$X(t) = \mu t + \sigma B(t)$$

- A continuous sample path process with independent increments
- $X(t+s) - X(s) \sim \mathrm{N}(\mu t, \sigma^2 t)$

$$\mathbb{P}\{X(t) \le y|X(0) = x\} = \Phi\left(\frac{y - \mu t - x}{\sigma\sqrt{t}}\right)$$

Absorption Probabilities

$$T_{ab} = \min\{t \ge 0; X(t) = a \text{ or } X(t) = b\}$$

$$u(x) = \mathbb{P}\{X(T_{ab}) = b|X(0) = x\}$$

We assume that absorption has not occurred at $(\Delta t, x + \Delta X)$

$$u(x) = \mathbb{E}(u(x + \Delta X))$$

Taylor series expansion of right hand side

$$\begin{aligned}u(x) &= \mathbb{E}\left[u(x) + u'(x)\Delta(X) + \frac{1}{2}u''(x)(\Delta X)^2 + o\left((\Delta X)^2\right)\right]\\ &= u(x) + u'(x)\mathbb{E}\left[\Delta(X)\right] + \frac{1}{2}u''(x)\mathbb{E}\left[(\Delta X)^2\right] + \mathbb{E}\left[o\left((\Delta X)^2\right)\right]\end{aligned}$$

$$\mathbb{E}\left[\Delta(X)\right] = \mu\Delta t, \qquad \mathbb{E}\left[(\Delta X)^2\right] = \sigma^2\Delta t + (\mu\Delta t)^2$$

$$\mu u'(x) + \frac{1}{2}\sigma^2 u''(x) = 0, \qquad u(x) = Ae^{-\frac{2\mu x}{\sigma^2}} + B$$

Absorption probabilities Theorem 8.1

$$\mu u'(x) + \frac{1}{2}\sigma^2 u''(x) = 0, \qquad u(x) = Ae^{-\frac{2\mu x}{\sigma^2}} + B$$

$$u(x) = Ae^{-\frac{2\mu x}{\sigma^2}} + B, \qquad u(a) = 0, u(b) = 1$$

$$u(x) = \frac{e^{-\frac{2\mu x}{\sigma^2}} - e^{-\frac{2\mu a}{\sigma^2}}}{e^{-\frac{2\mu b}{\sigma^2}} - e^{-\frac{2\mu a}{\sigma^2}}}$$

Mean Time to Absorption

$$T_{ab} = \min\{t \geq 0; X(t) = a \text{ or } X(t) = b\}$$

$$v(x) = \mathbb{E}\{T_{ab} | X(0) = x\}$$

$$v(x) = \Delta t + \mathbb{E}(v(x + \Delta X))$$

$$1 + \mu v'(x) + \frac{1}{2}\sigma^2 v''(x) = 0$$

$$v(a) = v(b) = 0$$

$$v(x) = \frac{1}{\mu}\left[u(x)(b-a) - (x-a)\right]$$

Maximum of Brownian Motion with Negative Drift

We assume $\mu < 0$, using Theorem 8.2.1

$$u(x) = \frac{e^{-\frac{2\mu x}{\sigma^2}} - e^{-\frac{2\mu a}{\sigma^2}}}{e^{-\frac{2\mu b}{\sigma^2}} - e^{-\frac{2\mu a}{\sigma^2}}}$$

we get

$$\mathbb{P}\{\max_{0 \leq t} X(t) > y\} = \lim_{a \to -\infty} \frac{e^{-\frac{2\mu \cdot 0}{\sigma^2}} - e^{-\frac{2\mu a}{\sigma^2}}}{e^{-\frac{2\mu y}{\sigma^2}} - e^{-\frac{2\mu a}{\sigma^2}}} = e^{-\frac{2|\mu|}{\sigma^2}y}$$

Geometric Brownian Motion

$$Z(t) = ze^{\left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma B(t)} \qquad (Z(0) = z)$$

$$X(t) = \log(z) + \log(Z(t), \qquad X(t) \sim \mathrm{N}\left(\alpha - \frac{1}{2}\sigma^2, \sigma^2\right)$$

$$\mathbb{E}(Z(t)) = ze^{\left(\alpha - \frac{1}{2}\sigma^2\right)t}\mathbb{E}\left[e^{\sigma B(t)}\right] = ze^{\left(\alpha - \frac{1}{2}\sigma^2\right)t}e^{\left(\frac{1}{2}\sigma^2\right)t} = ze^{\alpha t}$$

For $\alpha < \frac{1}{2}\sigma^2$ we have that $Z(t) \to 0$ with probability 1

$$\mathbb{V}\mathrm{ar}(Z(t)) = z^2 e^{2\alpha t}\left(e^{\sigma^2 t} - 1\right)$$