Brownian Motion

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Brownian Motion

Today:

- Definition and first properties
- Reflection principle and maximum variable

Next week

- Derived processes
- Brownian motion with drift

Two weeks from now

- Ohrnstein-Uhlenbeck process

Brownian Motion: Definition

Definition

Brownian motion with diffusion coefficient σ^2 is a stochastic process $\{B(t); t \geq 0\}$ with the properties

(a) *Every increment $B(s+t) - B(s)$ is normally distributed with mean zero and variance $\sigma^2 t$; $\sigma^2 > 0$ is a fixed parameter*

(b) *For every pair of disjoint time intervals $(t_1, t_2], (t_3, t_4]$, with $0 \leq t_1 < t_2 \leq t_3 < t_4$, the increments $B(t_4) - B(t_3)$ and $B(t_2) - B(t_1)$ are independent random variables and similarly for n disjoint time intervals, where n is an arbitrary positive integer.*

(c) *$B(0) = 0$, and $B(t)$ is a continuous function of t*

Diffusion equation

$$\frac{\partial p}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 p}{\partial y^2}$$

$$p(y,t|x) = \frac{1}{\sqrt{2\pi t\sigma}} e^{-\frac{(y-x)^2}{2t\sigma^2}}$$

$$\frac{\partial p}{\partial t} = \frac{1}{\sqrt{2\pi\sigma}}\left(-\frac{1}{2t\sqrt{t}}\right) e^{-\frac{(y-x)^2}{2t\sigma^2}} + \frac{1}{\sqrt{2\pi t\sigma}} e^{-\frac{(y-x)^2}{2t\sigma^2}} \frac{(y-x)^2}{2t^2\sigma^2}$$

$$\frac{\partial p}{\partial y} = \frac{1}{\sqrt{2\pi t\sigma}} e^{-\frac{(y-x)^2}{2t\sigma^2}} \left(\frac{-(y-x)}{t\sigma^2}\right)$$

Standard Brownian motion: $\sigma^2 = 1$.

$$\phi_t(x) = \frac{1}{\sqrt{t}}\phi\left(\frac{x}{\sqrt{t}}\right), \qquad \Phi_t(x) = \Phi\left(\frac{x}{\sqrt{t}}\right)$$

Covariance Function

$$\begin{aligned}\mathbb{C}\text{ov}[B(s),B(t)] &= \mathbb{E}[B(s)B(t)] \\ &= \mathbb{E}[B(s)\,(B(t)-B(s)+B(s))] \\ &= \mathbb{E}[B(s)^2]+\mathbb{E}[B(s)]\mathbb{E}[B(t)-B(s)] \\ &= s\sigma^2\end{aligned}$$

Invariance Principle

Let ξ_i be i.i.d. with $\mathbb{E}(\xi_i)=0$ and $\mathbb{V}\text{ar}(x_i)=1$; then

$$S_n = \xi_1+\cdots+\xi_n$$

$$\lim_{n\to\infty}\mathbb{P}\left\{\frac{S_n}{\sqrt{n}}\le x\right\}=\Phi(x), \qquad \text{CLT}$$

$$B_n(t)=\frac{S_{[nt]}}{\sqrt{n}}$$

$$B_n(t)=\frac{S_k}{\sqrt{n}}=\frac{S_k}{\sqrt{k}}\frac{\sqrt{k}}{\sqrt{n}}, \qquad \text{for } [nt]\le k<[nt]+1$$

Or

$$B_n(t)=\frac{S_k}{\sqrt{n}}=\frac{S_k}{\sqrt{k}}\frac{\sqrt{k}}{\sqrt{n}}, \qquad \text{for } \frac{k}{n}\le t<\frac{k}{n}+1$$

The normalized sum should show Brownian behaviour for n large

Gaussian Processes

A random vector $(X_1,\ldots,X_n)$ is said to be multivariate normal iff $Y=\alpha_1X_1+\ldots\alpha_nX_n$ is univariate Gaussian for all real α_i
With $\mu_i=\mathbb{E}(X_i)$ and $\Gamma_{ij}=\mathbb{C}\text{ov}(X_i,X_j)$ we get

$$f(x_1,\ldots,x_n)=f(\boldsymbol{x})=\frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\text{Det}(\boldsymbol{\Gamma})}}e^{-\frac{1}{2}\boldsymbol{x}'\boldsymbol{\Gamma}^{-1}\boldsymbol{x}}$$

$$\boldsymbol{x},\boldsymbol{X}$$

Gaussian Process

$$\mu(t)=\mathbb{E}[X(t)],\Gamma(s,t)=\mathbb{E}[\{X(s)-\mu(s)\}\{X(t)-\mu(t)\}]$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j\Gamma(t_i,t_j)\ge 0$$

The Reflection Principle

$$\tau=\min\{u\ge 0; B(u)=x\}$$

$$B^*(u)=\begin{cases} B(u) & \text{for } u\le\tau, \\ x-[B(u)-x] & \text{for } \tau\le u\end{cases}$$

$$\mathbb{P}\left\{\max_{0\le u\le t}B(u)>x\right\}=2\mathbb{P}(B(t)>x)$$

$$M(t)=\max_{0\le u\le t}B(u)$$

$$\mathbb{P}(M(t)>x)=2[1-\Phi_t(x)]$$

Time to First Reach a Level

$$
\begin{aligned}
\tau_x &= \min\{u \geq 0; B(u) = x\} \\
\mathbb{P}(\tau_x \leq t) &= \mathbb{P}(M(t) > x) \\
2[1 - \Phi_t(x)] &= \frac{2}{\sqrt{2\pi t}} \int_x^{\infty} e^{-\xi^2/(2t)} \mathrm{d}\xi \\
&= \sqrt{\frac{2}{\pi}} \int_{\frac{x}{\sqrt{t}}}^{\infty} e^{-\eta^2/(2)} \mathrm{d}\eta \\
f_{\tau_x}(t) &= \frac{1}{\sqrt{2\pi}} \frac{x}{t\sqrt{t}} e^{-x^2/(2t)}
\end{aligned}
$$