

Brownian Motion

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Today:

- ▶ Definition and first properties
- ▶ Reflection principle and maximum variable

Next week

- ▶ Derived processes
- ▶ Brownian motion with drift

Two weeks from now

- ▶ Ornstein-Uhlenbeck process

Brownian Motion: Definition

Definition

Brownian motion with diffusion coefficient σ^2 is a stochastic process $\{B(t); t \geq 0\}$ with the properties

- (a) Every increment $B(s + t) - B(s)$ is normally distributed with mean zero and variance $\sigma^2 t$; $\sigma^2 > 0$ is a fixed parameter*
- (b) For every pair of disjoint time intervals $(t_1, t_2]$, $(t_3, t_4]$, with $0 \leq t_1 < t_2 \leq t_3 < t_4$, the increments $B(t_4) - B(t_3)$ and $B(t_2) - B(t_1)$ are independent random variables and similarly for n disjoint time intervals, where n is an arbitrary positive integer.*
- (c) $B(0) = 0$, and $B(t)$ is a continuous function of t*

Diffusion equation

$$\frac{\partial p}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 p}{\partial y^2}$$

$$p(y, t|x) = \frac{1}{\sqrt{2\pi t\sigma}} e^{-\frac{(y-x)^2}{2t\sigma^2}}$$

$$\frac{\partial p}{\partial t} = \frac{1}{\sqrt{2\pi\sigma}} \left(-\frac{1}{2t\sqrt{t}} \right) e^{-\frac{(y-x)^2}{2t\sigma^2}} + \frac{1}{\sqrt{2\pi t\sigma}} e^{-\frac{(y-x)^2}{2t\sigma^2}} \frac{(y-x)^2}{2t^2\sigma^2}$$

$$\frac{\partial p}{\partial y} = \frac{1}{\sqrt{2\pi t\sigma}} e^{-\frac{(y-x)^2}{2t\sigma^2}} \left(\frac{-(y-x)}{t\sigma^2} \right)$$

Standard Brownian motion: $\sigma^2 = 1$.

$$\phi_t(x) = \frac{1}{\sqrt{t}} \phi\left(\frac{x}{\sqrt{t}}\right), \quad \Phi_t(x) = \Phi\left(\frac{x}{\sqrt{t}}\right)$$

Covariance Function

$$\begin{aligned}\text{Cov}[B(s), B(t)] &= \mathbb{E}[B(s)B(t)] \\ &= \mathbb{E}[B(s)(B(t) - B(s) + B(s))] \\ &= \mathbb{E}[B(s)^2] + \mathbb{E}[B(s)]\mathbb{E}[B(t) - B(s)] \\ &= s\sigma^2\end{aligned}$$

Invariance Principle

Let ξ_i be i.i.d. with $\mathbb{E}(\xi_i) = 0$ and $\text{Var}(\xi_i) = 1$; then

$$S_n = \xi_1 + \cdots + \xi_n$$

$$\lim_{n \rightarrow \infty} \mathbb{P} \left\{ \frac{S_n}{\sqrt{n}} \leq x \right\} = \Phi(x), \quad \text{CLT}$$

$$B_n(t) = \frac{S_{[nt]}}{\sqrt{n}}$$

$$B_n(t) = \frac{S_k}{\sqrt{n}} = \frac{S_k}{\sqrt{k}} \frac{\sqrt{k}}{\sqrt{n}}, \quad \text{for } [nt] \leq k < [nt] + 1$$

Or

$$B_n(t) = \frac{S_k}{\sqrt{n}} = \frac{S_k}{\sqrt{k}} \frac{\sqrt{k}}{\sqrt{n}}, \quad \text{for } \frac{k}{n} \leq t < \frac{k}{n} + 1$$

The normalized sum should show Brownian behaviour for n large

Gaussian Processes

A random vector (X_1, \dots, X_n) is said to be multivariate normal iff $Y = \alpha_1 X_1 + \dots + \alpha_n X_n$ is univariate Gaussian for all real α_j .
With $\mu_j = \mathbb{E}(X_j)$ and $\Gamma_{ij} = \text{Cov}(X_i, X_j)$ we get

$$f(x_1, \dots, x_n) = f(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{n}{2}} \sqrt{\text{Det}(\Gamma)}} e^{-\frac{1}{2} \mathbf{x}' \Gamma^{-1} \mathbf{x}}$$

\mathbf{x}, \mathbf{X}

Gaussian Process

$$\mu(t) = \mathbb{E}[X(t)], \Gamma(s, t) = \mathbb{E}[\{X(s) - \mu(s)\}\{X(t) - \mu(t)\}]$$

$$\sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j \Gamma(t_i, t_j) \geq 0$$

The Reflection Principle

$$\tau = \min\{u \geq 0; B(u) = x\}$$

$$B^*(u) = \begin{cases} B(u) & \text{for } u \leq \tau, \\ x - [B(u) - x] & \text{for } \tau \leq u \end{cases}$$

$$\mathbb{P} \left\{ \max_{0 \leq u \leq t} B(u) > x \right\} = 2\mathbb{P}(B(t) > x)$$

$$M(t) = \max_{0 \leq u \leq t} B(u)$$

$$\mathbb{P}(M(t) > x) = 2[1 - \Phi_t(x)]$$

Time to First Reach a Level

$$\begin{aligned}\tau_x &= \min\{u \geq 0; B(u) = x\} \\ \mathbb{P}(\tau_x \leq t) &= \mathbb{P}(M(t) > x) \\ 2[1 - \Phi_t(x)] &= \frac{2}{\sqrt{2\pi t}} \int_x^\infty e^{-\xi^2/(2t)} d\xi \\ &= \sqrt{\frac{2}{\pi}} \int_{\frac{x}{\sqrt{t}}}^\infty e^{-\eta^2/(2)} d\eta \\ f_{\tau_x}(t) &= \frac{1}{\sqrt{2\pi}} \frac{x}{t\sqrt{t}} e^{-x^2/(2t)}\end{aligned}$$