Discrete Time Markov Chains, Definition and classification

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Discrete time Markov chains

Today:

- Short recap of probability theory
- Markov chain introduction (Markov property)
- Chapmann-Kolmogorov equations
- First step analysis

Next week

- Random walks
- First step analysis revisited
- Branching processes
- Generating functions

Two weeks from now

- Classification of states
- Classification of chains
- Discrete time Markov chains - invariant probability distribution

Basic concepts in probability

Sample space	Ω	set of all possible outcomes
Outcome	ω	
Event	A, B	
Complementary event	$A^c = \Omega \backslash A$	
Union	$A \cup B$	outcome in at least one of A or B
Intersection	$A \cap B$	Outcome is in both A and B
(Empty) or impossible event	$\emptyset$	

Probability axioms and first results

$$0 \leq \mathbb{P}(A) \leq 1, \qquad \mathbb{P}(\Omega) = 1$$

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) \qquad \text{for } A \cap B = \emptyset$$

Leading to

$$\mathbb{P}(\emptyset) = 0, \qquad \mathbb{P}(A^c) = 1 - \mathbb{P}(A)$$

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

(inclusion- exlusion)

Conditional probability and independence

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \Leftrightarrow \mathbb{P}(A \cap B) = \mathbb{P}(A|B)\mathbb{P}(B)$$

(multiplication rule)

$$\cup_i B_i = \Omega \qquad B_i \cap B_j = \emptyset \qquad i \neq j$$

$$\mathbb{P}(A) = \sum_i \mathbb{P}(A|B_i)\mathbb{P}(B_i) \qquad \text{(law of total probability)}$$

Independence:

$$\mathbb{P}(A|B) = \mathbb{P}(A|B^c) = \mathbb{P}(A) \Leftrightarrow \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

Discrete random variables

Mapping from sample space to metric space
(Read: Real space)

Probability mass function

$$f(x) = \mathbb{P}(X = x) = \mathbb{P}\left(\{\omega | X(\omega) = x\}\right)$$

Distribution function

$$F(x) = \mathbb{P}(X \leq x) = \mathbb{P}\left(\{\omega | X(\omega) \leq x\}\right) = \sum_{t \leq x} f(t)$$

Expectation

$$\mathbb{E}(X) = \sum_x x\mathbb{P}(X = x), \quad \mathbb{E}(g(X)) = \sum_x g(x)\mathbb{P}(X = x) = \sum_x g(x)f(x)$$

Joint distribution

$$f(x_1, x_2) = \mathbb{P}(X_1 = x_1, X_2 = x_2), \qquad F(x_1, x_2) = \mathbb{P}(X_1 \leq x_1, X_2 \leq x_2)$$

$$f_{X_1}(x_1) = \mathbb{P}(X_1 = x_1) = \sum_{x_2} \mathbb{P}(X_1 = x_1, X_2 = x_2) = \sum_{x_2} f(x_1, x_2)$$

$$F_{X_1}(x_1) = \sum_{t \leq x_1, x_2} \mathbb{P}(X_1 = t_1, X_2 = x_2) = F(x_1, \infty)$$

Straightforward to extend to n variables

We can define the joint distribution of (X_0, X_1) through

$$\mathbb{P}(X_0 = x_0, X_1 = x_1) = \mathbb{P}(X_0 = x_0)\mathbb{P}(X_1 = x_1 | X_0 = x_0) = \mathbb{P}(X_0 = x_0) P_{x_0,x_1}$$

Suppose now some stationarity in addition that X_2 conditioned on X_1 is independent on X_0

$$\begin{aligned}
&\mathbb{P}(X_0 = x_0, X_1 = x_1, X_2 = x_2) = \\
&\mathbb{P}(X_0 = x_0)\mathbb{P}(X_1 = x_1 | X_0 = x_0)\mathbb{P}(X_2 = x_2 | X_0 = x_0, X_1 = x_1) = \\
&\mathbb{P}(X_0 = x_0)\mathbb{P}(X_1 = x_1 | X_0 = x_0)\mathbb{P}(X_2 = x_2 | X_1 = x_1) = \\
&p_{x_0} P_{x_0,x_1} P_{x_1,x_2}
\end{aligned}$$

which generalizes to arbitrary n.

Markov property

$$\mathbb{P}(X_n = x_n | H)$$

$$= \mathbb{P}(X_n = x_n | X_0 = x_0, X_1 = x_1, X_2 = x_2, \ldots X_{n-1} = x_{n-1})$$
$$= \mathbb{P}(X_n = x_n | X_{n-1} = x_{n-1})$$

- Generally the next state depends on the current state and the time
- In most applications the chain is assumed to be time homogeneous, i.e. it does not depend on time
- The only parameters needed are $\mathbb{P}(X_n = j | X_{n-1} = i) = p_{ij}$
- We collect these parameters in a matrix $\boldsymbol{P} = \{p_{ij}\}$
- The joint probability of the first n occurrences is

$$\mathbb{P}(X_0 = x_0, X_1 = x_1, X_2 = x_2 \ldots, X_n = x_n) = p_{x_0} P_{x_0,x_1} P_{x_1,x_2} \ldots P_{x_{n-1},x_n}$$

A profuse number of applications

- Storage/inventory models
- Telecommunications systems
- Biological models
- X_n the value attained at time n
- X_n could be
  - The number of cars in stock
  - The number of days since last rainfall
  - The number of passengers booked for a flight
  - See textbook for further examples

Example 1: Random walk with two reflecting barriers 0 and N

$$
\boldsymbol{P} = \left\| \begin{array}{ccccccc}
1-p & p & 0 & \dots & 0 & 0 & 0 \\
q & 1-p-q & p & \dots & 0 & 0 & 0 \\
0 & q & 1-p-q & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & q & 1-p-q & p \\
0 & 0 & 0 & \dots & 0 & q & 1-q
\end{array} \right\|
$$

Example 2: Random walk with one reflecting barrier at 0

$$\boldsymbol{P} = \left\|\begin{array}{cccccc} 1-p & p & 0 & 0 & 0 & \dots \\ q & 1-p-q & p & 0 & 0 & \dots \\ 0 & q & 1-p-q & p & 0 & \dots \\ 0 & 0 & q & 1-p-q & p & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{array}\right.$$

Example 3: Random walk with two absorbing barriers

$$\boldsymbol{P} = \begin{Vmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ q & 1-p-q & p & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & q & 1-p-q & p & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & q & 1-p-q & p & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & q & 1-p-q & p \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{Vmatrix}$$

The matrix can be finite (if the Markov chain is finite)

$$\boldsymbol{P} = \left\|\begin{array}{ccccc} p_{1,1} & p_{1,2} & p_{1,3} & \cdots & p_{1,n} \\ p_{2,1} & p_{2,2} & p_{2,3} & \cdots & p_{2,n} \\ p_{3,1} & p_{3,2} & p_{3,3} & \cdots & p_{3,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{n,1} & p_{n,2} & p_{n,3} & \cdots & p_{n,n} \end{array}\right\|$$

Two reflecting/absorbing barriers

or infinite (if the Markov chain is infinite)

$$\boldsymbol{P} = \left\| \begin{matrix} p_{1,1} & p_{1,2} & p_{1,3} & \cdots & p_{1,n} & \cdots \\ p_{2,1} & p_{2,2} & p_{2,3} & \cdots & p_{2,n} & \cdots \\ p_{3,1} & p_{3,2} & p_{3,3} & \cdots & p_{3,n} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{n,1} & p_{n,2} & p_{n,3} & \cdots & p_{n,n} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{matrix} \right.$$

At most one barrier

The matrix P can be interpreted as

- the engine that drives the process
- the statistical descriptor of the quantitative behaviour
- a collection of discrete probability distributions
  - For each i we have a conditional distribution
  - What is the probability of the next state being j knowing that the current state is i $p_{ij} = \mathbb{P}\left(X_n = j | X_{n-1} = i\right)$
  - $\sum_j p_{ij} = 1$
  - We say that $\boldsymbol{P}$ is a stochastic matrix

More definitions and the first properties

- We have defined rules for the behaviour from one value and onwards
- Boundary conditions specify e.g. behaviour of X_0
  - X_0 could be certain $X_0 = a$
  - or random $\mathbb{P}(X_0 = i) = p_i$
  - Once again we collect the possibly infinite many parameters in a vector $\boldsymbol{p}$

n step transition probabilities

$$\mathbb{P}\left(X_n = j | X_0 = i\right) = P_{ij}^{(n)}$$

- the probability of being in j at the n'th transition having started in i
- Once again collected in a matrix $\boldsymbol{P}^{(n)} = \{P_{ij}^{(n)}\}$
- The rows of $\boldsymbol{P}^{(n)}$ can be interpreted like the rows of $\boldsymbol{P}$
- We can define a new Markov chain on a larger time scale ($\boldsymbol{P}^n$)

Small example

$$\boldsymbol{P} = \left\| \begin{array}{cccc} 1-p & p & 0 & 0 \\ q & 0 & p & 0 \\ 0 & q & 0 & p \\ 0 & 0 & q & 1-q \end{array} \right\|$$

$$\boldsymbol{P}^{(2)} = \left\| \begin{array}{cccc} (1-p)^2+pq & (1-p)p & p^2 & 0 \\ q(1-p) & 2qp & 0 & p^2 \\ q^2 & 0 & 2qp & p(1-q) \\ 0 & q^2 & (1-q)q & (1-q)^2+qp \end{array} \right\|$$

Chapmann Kolmogorov equations

- There is a generalisation of the example above
- Suppose we start in i at time 0 and wants to get to j at time $n+m$
- At some intermediate time n we must be in some state k
- We apply the law of total probability $\mathbb{P}(B) = \sum_k \mathbb{P}(B|A_k)\mathbb{P}(A_k)$

$$\mathbb{P}(X_{n+m} = j | X_0 = i)$$

$$= \sum_k \mathbb{P}(X_{n+m} = j | X_0 = i, X_n = k)\mathbb{P}(X_n = k | X_0 = i)$$

$$\sum_k \mathbb{P}\left(X_{n+m} = j | X_0 = i, X_n = k\right) \mathbb{P}\left(X_n = k | X_0 = i\right)$$

by the Markov property we get

$$\sum_k \mathbb{P}\left(X_{n+m} = j | X_n = k\right) \mathbb{P}\left(X_n = k | X_0 = i\right)$$

$$= \sum_k P_{kj}^{(m)} P_{ik}^{(n)} = \sum_k P_{ik}^{(n)} P_{kj}^{(m)}$$

which in matrix formulation is

$$\boldsymbol{P}^{(n+m)} = \boldsymbol{P}^{(n)} \boldsymbol{P}^{(m)} = \boldsymbol{P}^{n+m}$$

The probability of X_n

- The behaviour of the process itself - X_n
- The behaviour conditional on $X_0 = i$ is known ($P_{ij}^{(n)}$)
- Define $\mathbb{P}(X_n = j) = p_j^{(n)}$
- with $\boldsymbol{p}^{(n)} = \{p_j^{(n)}\}$ we find

$$\boldsymbol{p}^{(n)} = \boldsymbol{p}\boldsymbol{P}^{(n)} = \boldsymbol{p}\boldsymbol{P}^{n}$$

Small example - revisited

$$\boldsymbol{P} = \left|\left|\begin{array}{cccc} 1-p & p & 0 & 0 \\ q & 0 & p & 0 \\ 0 & q & 0 & p \\ 0 & 0 & q & 1-q \end{array}\right|\right|$$

with $\boldsymbol{p} = \left(\frac{1}{3}, 0, 0, \frac{2}{3}\right)$ we get

$$\boldsymbol{p}^{(1)} = \left(\frac{1}{3}, 0, 0, \frac{2}{3}\right) \left|\left|\begin{array}{cccc} 1-p & p & 0 & 0 \\ q & 0 & p & 0 \\ 0 & q & 0 & p \\ 0 & 0 & q & 1-q \end{array}\right|\right| = \left(\frac{1-p}{3}, \frac{p}{3}, \frac{2q}{3}, \frac{2(1-q)}{3}\right)$$

$$\boldsymbol{p} = \left(\frac{1}{3}, 0, 0, \frac{2}{3}\right),$$

$$\boldsymbol{P}^2 = \left\| \begin{array}{cccc} (1-p)^2 + pq & (1-p)p & p^2 & 0 \\ q(1-p) & 2qp & 0 & p^2 \\ q^2 & 0 & 2qp & p(1-q) \\ 0 & q^2 & (1-q)q & (1-q)^2 + qp \end{array} \right\|$$

$$\boldsymbol{p}^{(2)} = \left(\frac{1}{3}, 0, 0, \frac{2}{3}\right) \cdot$$

$$\left\|\begin{array}{cccc} (1-p)^2+pq & (1-p)p & p^2 & 0 \\ q(1-p) & 2qp & 0 & p^2 \\ q^2 & 0 & 2qp & p(1-q) \\ 0 & q^2 & (1-q)q & (1-q)^2+qp \end{array}\right\|$$

$$= \left(\frac{(1-p)^2+pq}{3}, \frac{(1-p)p}{3}, \frac{4qp}{3}, \frac{2p(1-q)}{3}\right)$$

First step analysis - setup

Consider the transition probability matrix

$$\boldsymbol{P} = \left\| \begin{array}{ccc} 1 & 0 & 0 \\ \alpha & \beta & \gamma \\ 0 & 0 & 1 \end{array} \right\|$$

Define

$$T = \min\{n \geq 0 : X_n = 0 \text{ or } X_n = 2\}$$

and

$$u = \mathbb{P}(X_T = 0 | X_0 = 1) \qquad v = \mathbb{E}(T | X_0 = 1)$$

First step analysis - absorption probability

$$\begin{aligned}
u &= \mathbb{P}(X_T = 0 | X_0 = 1) \\
&= \sum_{k=0}^{2} \mathbb{P}(X_1 = k | X_0 = 1) \mathbb{P}(X_T = 0 | X_0 = 1, X_1 = k) \\
&= \sum_{k=0}^{2} \mathbb{P}(X_1 = k | X_0 = 1) \mathbb{P}(X_T = 0 | X_1 = k) \\
&= P_{1,0} \cdot 1 + P_{1,1} \cdot u + P_{1,2} \cdot 0.
\end{aligned}$$

And we find

$$u = \frac{P_{1,0}}{1 - P_{1,1}} = \frac{\alpha}{\alpha + \gamma}$$

First step analysis - time to absorption

$$
\begin{aligned}
v &= \mathbb{E}(T|X_0 = 1) \\
&= \sum_{k=0}^{2} \mathbb{P}(X_1 = k|X_0 = 1)\mathbb{E}(T|X_0 = 1, X_1 = k) \\
&= 1 + \sum_{k=0}^{2} \mathbb{P}(X_1 = k|X_0 = 1)\mathbb{E}(T|X_0 = k)(\text{ NB!} \\
&= 1 + P_{1,0} \cdot 0 + P_{1,1} \cdot v + P_{1,2} \cdot 0.
\end{aligned}
$$

And we find

$$
v = \frac{1}{1 - P_{1,1}} = \frac{1}{1 - \beta}
$$

More than one transient state

$$\boldsymbol{P} = \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ P_{1,0} & P_{1,1} & P_{1,2} & P_{1,3} \\ P_{2,0} & P_{2,1} & P_{2,2} & P_{2,3} \\ 0 & 0 & 0 & 1 \end{array} \right\|$$

- Here we will need conditional probabilities $u_i = \mathbb{P}(X_T = 0 | X_0 = i)$
- and conditional expectations $v_i = \mathbb{E}(T | X_0 = i)$

Leading to

$$
\begin{aligned}
u_1 &= P_{1,0} + P_{1,1}u_1 + P_{1,2}u_2 \\
u_2 &= P_{2,0} + P_{2,1}u_1 + P_{2,2}u_2
\end{aligned}
$$

and

$$
\begin{aligned}
v_1 &= 1 + P_{1,1}v_1 + P_{1,2}v_2 \\
v_2 &= 1 + P_{2,1}v_1 + P_{2,2}v_2
\end{aligned}
$$

General finite state Markov chain

$$\boldsymbol{P} = \left\| \begin{array}{cc} \boldsymbol{Q} & \boldsymbol{R} \\ \boldsymbol{0} & \boldsymbol{I} \end{array} \right\|$$

General absorbing Markov chain

$$T = \min\{n \geq 0, X_n \geq r\}$$

In state j we accumulate reward $g(j)$, w_i is expected total reward conditioned on start in state i

$$w_i = \mathbb{E}\left(\sum_{n=0}^{T-1} g(X_n) | X_0 = i\right)$$

leading to

$$w_i = g(i) + \sum_j P_{i,j} w_j$$

Special cases of general absorbing Markov chain

- $g(i) = 1$ expected time to absorption (v_i)
- $g(i) = \delta_{ik}$ expected visits to state k before absorption