External Memory I

- Computational Models
- Scanning
- Sorting
- Searching

Philip Bille

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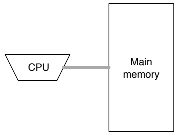
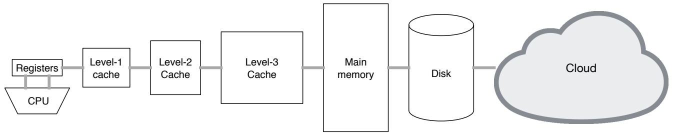
Computational Models



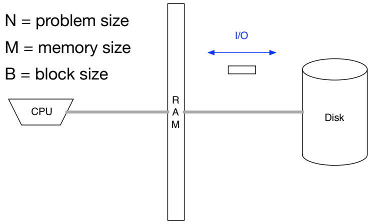
- (word) RAM Model
  - Infinite memory of w-bit memory cells
  - Instructions: Memory access, arithmetic operations, boolean operations, control-flow operations, etc.
- Complexity model.
  - Time = number of instructions.
  - Space = number of memory cells used.

Computational Models



- Macbook Pro (late 2023)
  - CPU: 16 Core M3 Max
  - Registers: ?
  - L1 cache: 320 KB per core
  - L2 cache: 32 MB per core
  - Memory: 128 GB
  - Disk: 1 TB
  - Instructions: Memory access, arithmetic operations, boolean operations, control-flow operations, etc.
- Complexity?

Computational Models



- I/O model [Aggarwal and Vitter 1988].
  - Limited memory, Infinite disk
  - Instructions: Disk I/O operations, memory access, arithmetic operations, boolean operations, control-flow operations, etc.
- Complexity model.
  - I/Os = Number of disk I/Os
  - Computation is free (!)

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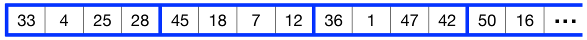
Scanning



33	4	25	28	45	18	7	12	36	1	47	42	50	16	...

- Scanning. Given an array A of N values (stored in N/B blocks), process all values from left-to-right.
- I/Os. O(N/B).

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Sorting

33	4	25	28	45	18	7	12	36	1	47	42	50	16	31

1	4	7	12	16	18	25	28	31	33	36	42	45	47	50

- Sorting. Given array A of N values (stored in N/B blocks), output the values in increasing order.

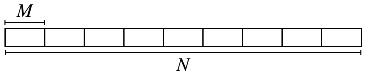
Sorting

- Which solutions do we know (on the RAM model)?

External Merge Sort

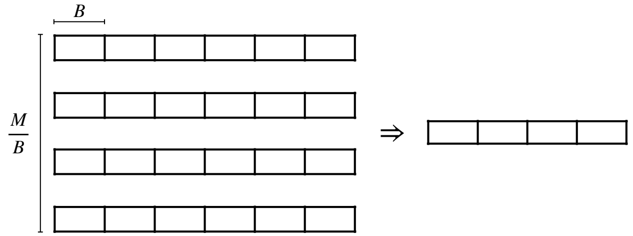
- Goal. Sorting in O(N/B $\log_{M/B}$ (N/B)) I/Os.
- Solution in 3 steps.
  - Base case.
  - External multi-way merge.
  - External merge sort.

External Merge Sort



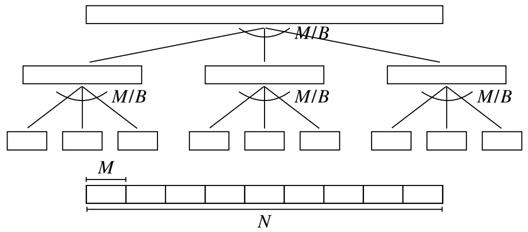
- Base case.
  - Partition N elements into N/M arrays of size M.
  - Load each into memory and sort.
- I/Os. O(N/B)

External Merge Sort



- Multiway merge algorithm.
  - Input is N elements in M/B arrays.
  - Load M/B first blocks into memory and sort.
  - Output B smallest elements.
  - Load more blocks into memory if needed.
  - Repeat.
- I/Os. O(N/B).

External Merge Sort



- Algorithm.
  - Partition N elements into N/M arrays of size M. Load each into memory and sort.
  - Apply M/B way external multiway merge until left with single sorted array.
- I/Os.
  - Sort N/M arrays: O(N/B) I/Os
  - Height of tree O($\log_{M/B}$(N/M))
  - Cost per level: O(N/B) I/Os.

Total I/Os: $O\left(\frac{N}{B}\log_{M/B}\frac{N}{M}\right) = O\left(\frac{N}{B}\log_{M/B}\frac{N}{B}\right)$

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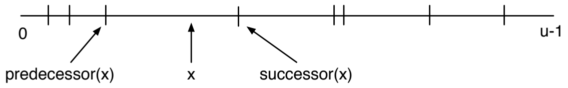
Searching

- Searching. Maintain a set S ⊆ U = {0, ..., u-1} supporting
  - member(x): determine if x ∈ S
  - predecessor(x): return largest element in S ≤ x.
  - successor(x): return smallest element in S ≥ x.
  - insert(x): set S = S ∪ {x}
  - delete(x): set S = S - {x}



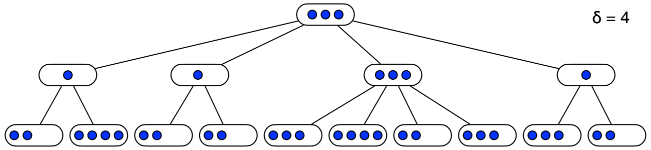
Searching

- Applications.
  - Relational data bases.
  - File systems.

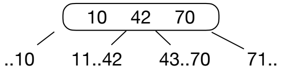
Searching

- Which solutions do we know (on the RAM model)?

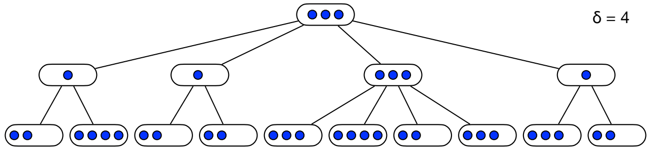
B-tree



- B-tree of order $\delta = \Theta(B)$ with N keys.
  - Keys in leaves. Routing elements in internal nodes.
  - Degree between $\delta/2$ and δ.
  - Root degree between 2 and δ.
  - Leaves store between $\delta/2$ and δ keys.
  - All leaves have the same depth.
- Height. $\Theta(\log_\delta (N/B)) = \Theta(\log_B N)$

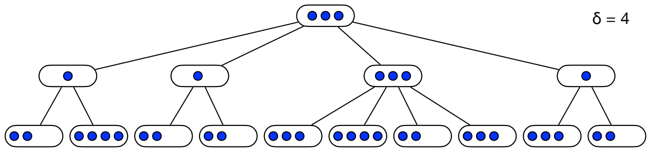


B-tree

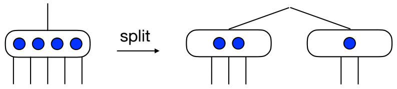


- Searching.
  - Find leaf using routing elements.
- I/Os. $O(\log_B N)$.

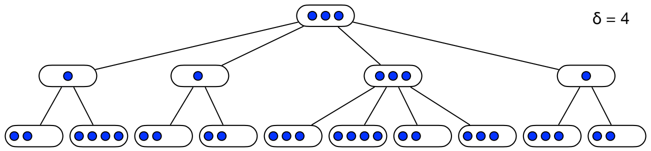
B-tree



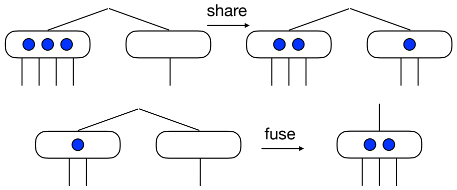
- Insertion.
  - Find leaf.
  - Insert key.
  - Split nodes on path.
- I/Os. O(log$_B$ N).



B-tree



- Deletion.
  - Find leaf.
  - Delete key.
  - Share or fuse nodes on path.
- I/Os. O(log$_B$ N).



Basic Bounds

	Internal	External
Scanning	O(N)	scan(N) = O(N/B)
Sorting	O(Nlog N)	sort(N) = O((N/B)log$_{M/B}$ (N/B))
Searching	O(log N)	search(N) = O(log$_B$(N))

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