Degrees of Freedom

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This brief note is about an idea I had for computing the *Degrees of Freedom* of a geometrical primitive. I assume the method is known, but I have not found the method in literature. Probably because I did not look hard enough.

Actually, the idea was spawned when two friends (Bent and Theo) and myself were driving toward downtown San Francisco from Silicon Valley on US 101. It is not advisable to use US 101 for this kind of trip. Use 280 instead. Only use US 101 for short trips. Unsurprisingly, we were stuck in traffic, and Bent got bored with driving so he said "tell me something I don't know". Asked about what exactly we should tell him he said "something about computer graphics, something practical I can use".

A thing popped into my mind.

– A point in 3D has 3 Degrees of Freedom and so does a line. It is just like in 2D where a point and a line both have two Degrees of Freedom. It is pretty easy to see in 2D. A point is described by two variables (x and y) and obviously has two DoF. A line can be completely described by an angle and the distance to the origin.

I didn't really explain why a 3D line should have 3 DoF but did some hand waving. Some time later we had wormed our way into SF, bought some gas and we were heading toward Fisherman's Wharf. At this point, I was driving and Bent had time to think.

"I don't believe that a line has only 3 DoF in space" he said. He ventured an explanation, and, of course, he was right. A plane in space can be described by three independent variables. That is pretty easy to see: Any plane is completely described by a perpendicular vector from the origin to the plane. Such a vector has 3 DoF and hence the plane also has three Degrees of Freedom. If we have a line in space we can also describe it by a perpendicular vector from the origin, but it is not fully constrained. A perpendicular vector from the origin constrains the line to a plane and a point it must pass through, but that leaves one Degree of Freedom. An angle can be used to nail down the line.

I could see that a line has four Degrees of Freedom, and I was annoyed that I had been wrong. To avoid being wrong about Degrees of Freedom, I decided to find a method for computing how many Degrees of Freedom a geometrical primitive has. The method occurred to me the following day as we were walking around in Chinatown.

Take a line in 3D space. A line can be specified by two points. However, once the line is constrained by the points, the points can still move along the line. Thus, one Degree of Freedom is retained per point. Since a point in 3D has three Degrees of Freedom, the total number of Degrees of Freedom used to nail down the line is two times three minus two times one equal to four. The result is correct.

The same method can be used to compute the DoF of a plane. A plane is fully constrained by three points, but the points are free to roam the plane, and this motion has two DoF. Hence, the plane has $3 \times 3 - 3 \times 2 = 3$ DoF. The formula holds.

A different way of computing the Degrees of Freedom of a line is to specify the line as the intersection of two planes. Once the line has been found, the planes may rotate about the line without affecting it. This rotation can be specified by one independent variable per plane. Since the planes have 3 DoF, it is clear that the line has $2 \times 3 - 2 \times 1 = 4$ DoF as it should.

In summary, I have proposed a simple method for computing the Degrees of Freedom of geometrical primitives. Constrain the primitive (A) using other primitives with a known number of DoF. Add the DoF of the constraining primitives and subtract the DoFs retained by the constraining primitives once A has been specified. The result of this computation is the DoFs of A.

The method has been seen to work in a number of cases, but, of course, I have not proven that it works in general or under what conditions precisely.