# Relating Abstract Datatypes and Z-Schemata*

Hubert Baumeister

University of Munich, Institute of Computer Science,
Oettingenstr. 67, D-80358 Munich, Germany
baumeist@informatik.uni-muenchen.de

**Abstract.** In this paper we investigate formally the relationship between the notion of abstract datatypes in an arbitrary institution, found in algebraic specification languages like Clear, ASL and CASL; and the notion of schemata from the model-oriented specification language Z. To this end the institution $\mathcal{S}$ of the logic underlying Z is defined and a translation of Z-schemata to abstract datatypes over $\mathcal{S}$ is given. The notion of a schema is internal to the logic of Z and thus specification techniques of Z relying on the notion of a schema can only be applied in the context of Z. By translating Z-schemata to abstract datatypes these specification techniques can be transformed to specification techniques using abstract datatypes. Since the notion of abstract datatypes is institution independent, this results in a separation of these specification techniques from the specification language Z and allows them to be applied in the context of other, e.g. algebraic, specification languages.

## 1 Introduction

As already noted by Spivey [11], schema-types, as used in the model-oriented specification language Z, are closely related to many-sorted signatures; and schemata are related to the notion of abstract datatypes found in algebraic specification languages.

Z is a model-oriented specification language based on set-theory. In the model-oriented approach to the specification of software systems specifications are explicit system models constructed out of either abstract or concrete primitives. This is in contrast to the approach used with algebraic or property-oriented specification languages like CASL [9], which identifies the interface of a software module, consisting of sorts and functions, and states the properties of the interface components using first-order formulas.

Specifications written in Z are structured using schemata and operations on schemata. A schema denotes a set of bindings of the form $\{(x_1, v_1), \ldots, (x_n, v_n)\}$. Operations on schemata include restriction of the elements of a schema to those satisfying a formula; logical operations like negation, conjunction, disjunction and quantification; and renaming and hiding of the components of a schema. Schemata, and thus the structuring mechanism of Z, are elements of the logic used by Z. This, on one hand, has the advantage of using Z again to reason about the structure of a specification, but, on the other hand, has the disadvantage that development methods and theoretical results referring to the structure of specifications cannot be easily transfered to other specification languages based on different logics.

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In contrast, the structuring primitives of property-oriented specification languages can be formulated independent from the logic underlying the particular specification language. This is done by using the notion of an institution introduced by Goguen and Burstall [5] to formalize the informal notion of a logical system. The building blocks of specifications are abstract datatypes which consist of an interface and a class of possible implementations of that interface. Operations on abstract datatypes are the restriction of the implementations to those satisfying a set of formulas; the union of abstract datatypes; hiding, adding and renaming of interface components. What exactly constitutes the components of an interface and how they are interpreted in implementations depends on the institution underlying the specification language. For example, in the institution of equational logic the components of an interface are sorts and operations. The implementations interpret the sorts as sets and the operations as functions on these sets.

The goal of this paper is to formalize the relationship between schemata and abstract datatypes, and to show a correspondence between the operations on abstract datatypes and operations on schemata. This relationship can be used to transfer results and methods used from Z to property-oriented specification languages and vice versa. For example, the Z-style for the specification of sequential systems can be transfered to property-oriented specification languages [2]. Further, the correspondence between operations on abstract datatypes and operations on schema suggests new operations on abstract datatypes like negation and disjunction.

However, we cannot compare schemata with abstract datatypes in an arbitrary institution; instead, we have to define first an institution $\mathcal{S}$ which formalizes the notion of the set-theory used in Z, and then compare schemata with abstract datatypes in this institution. The definition of the institution $\mathcal{S}$ has the further advantage that it can be used to define a variant of the specification language CASL, CASL-$\mathcal{S}$, based on set-theory instead of order-sorted partial first-order logic. This is possible because the semantics of most of CASL is largely independent from a particular institution (cf. Mossakowski [8]).

## 2 Institutions and Abstract Datatypes

The notion of institutions is an attempt to formalize the informal notion of a logical system and was developed by Goguen and Burstall [5] as a means to define the semantics of the specification language Clear [3] independent from a particular logic.

**Definition 1 (Institution).** *An institution* $\mathcal{I} = \langle \textsc{Sign}_{\mathcal{I}}, \mathsf{Str}_{\mathcal{I}}, \mathsf{Sen}_{\mathcal{I}}, \models^{\mathcal{I}} \rangle$ *consists of*

- *a category of* signatures $\textsc{Sign}_{\mathcal{I}}$*,*
- *a functor* $\mathsf{Str}_{\mathcal{I}} : \textsc{Sign}_{\mathcal{I}}^{op} \to \textsc{Cat}$ *assigning to each signature* $\Sigma$ *the category of* $\Sigma$*-structures and to each signature morphism* $\sigma : \Sigma \to \Sigma'$ *the reduct functor* $\_|_{\sigma} : \mathsf{Str}_{\mathcal{I}}(\Sigma') \to \mathsf{Str}_{\mathcal{I}}(\Sigma)$*,*
- *a functor* $\mathsf{Sen}_{\mathcal{I}} : \textsc{Sign}_{\mathcal{I}} \to \textsc{Set}$ *assigning to each signature* $\Sigma$ *the set of* $\Sigma$-formulas *and to each signature morphism* $\sigma : \Sigma \to \Sigma'$ *a translation* $\overline{\sigma}$ *of* $\Sigma$*-formulas to* $\Sigma'$*-formulas, and*

– *a family of* satisfaction relations $\models^{\mathcal{I}}{}_{\Sigma} \subseteq \mathsf{Str}_{\mathcal{I}}(\Sigma) \times \mathsf{Sen}_{\mathcal{I}}(\Sigma)$ *for* $\Sigma \in \textsc{Sign}_{\mathcal{I}}$ *indicating whether a $\Sigma$-formula $\varphi$ is valid in a $\Sigma$-structure $m$, written $m \models^{\mathcal{I}}{}_{\Sigma} \varphi$ or for short $m \models^{\mathcal{I}} \varphi$,*

*such that the* satisfaction condition *holds: for all signature morphisms $\sigma : \Sigma \to \Sigma'$, formulas $\varphi \in \mathsf{Sen}_{\mathcal{I}}(\Sigma)$ and structures $m' \in \mathsf{Str}_{\mathcal{I}}(\Sigma')$ we have*

$$m'|_{\sigma} \models^{\mathcal{I}} \varphi \textit{ if and only if } m' \models^{\mathcal{I}} \overline{\sigma}(\varphi)$$

We may write $M \models^{\mathcal{I}} \varphi$ for a class of $\Sigma$-structures $M$ and a $\Sigma$-formula $\varphi$ instead of $\forall m \in M\colon\ m \models^{\mathcal{I}} \varphi$, and similar for $m \models^{\mathcal{I}} \Phi$ and $M \models^{\mathcal{I}} \Phi$ for a set of $\Sigma$-formulas $\Phi$ and a $\Sigma$-structure $m$.

Traditionally, an abstract datatype $(\Sigma, M)$ is a specification of a datatype in a software system. The signature $\Sigma$ defines the external interface as a collection of sort and function symbols and $M$ is a class of $\Sigma$-algebras considered admissible implementations of that datatype. In the context of an arbitrary institution $\mathcal{I}$ an *abstract datatype* is a pair $(\Sigma, M)$ where $\Sigma$ is an element of $\textsc{Sign}_{\mathcal{I}}$ and $M$ is a full subcategory of $\mathsf{Str}_{\mathcal{I}}(\Sigma)$.

The basic operations on abstract datatypes are $I_{\Phi}$ (impose), $D_{\sigma}$ (derive), $T_{\sigma}$ (translate) and $+$ (union) (cf. Sannella and Wirsing [10]):

*Impose* allows to impose additional requirements on an abstract datatype. The semantics of an expression $I_{\Phi}(\Sigma, M)$ is the abstract datatype $(\Sigma, M')$ where $M'$ consists of all $\Sigma$-structures $m$ in $M$ satisfying all formulas in $\Phi$, i.e.

$$I_{\Phi}(\Sigma, M) = (\Sigma, \{m \in M \mid m \models^{\mathcal{I}} \Phi\}).$$

The *translate* operation can be used to rename symbols in a signature but also to add new symbols to a signature. If $\sigma$ is a signature morphism from $\Sigma$ to $\Sigma'$ then the expression $T_{\sigma}(\Sigma, M)$ denotes an abstract datatype $(\Sigma', M')$ where $M'$ contains all $\Sigma'$-structures $m$ which are extensions of some $\Sigma$-structure $m$ in $M$, i.e.

$$T_{\sigma}(\Sigma, M) = (\Sigma', \{m' \in \mathsf{Str}_{\mathcal{I}}(\Sigma') \mid m'|_{\sigma} \in M\}).$$

The *derive* operation allows to hide parts of a signature. $D_{\sigma}(\Sigma', M')$ denotes the abstract datatype having as signature the domain of $\sigma$ and as models the translations of the models of $\textsc{Sp}$ by $\_|_{\sigma}$, i.e.

$$D_{\sigma}(\Sigma', M') = (\Sigma, \{m'|_{\sigma} \mid m' \in M'\}).$$

At last, the *union* operation is used to combine two specifications. Since for arbitrary institutions, the union of signatures is not defined, we have to require that both specifications have the same signature. To form the union of two specifications of different signatures $\Sigma_1$ and $\Sigma_2$ one has to provide a signature $\Sigma$ and signature morphisms $\sigma_1 : \Sigma_1 \to \Sigma$ and $\sigma_2 : \Sigma_2 \to \Sigma$ and write $T_{\sigma_1}\textsc{Sp}_1 + T_{\sigma_2}\textsc{Sp}_2$. The semantics of $(\Sigma, M_1) + (\Sigma, M_2)$ is the abstract datatype $(\Sigma, M')$ where $M'$ is the intersection $M_1$ and $M_2$, i.e.

$$(\Sigma, M_1) + (\Sigma, M_2) = (\Sigma, M_1 \cap M_2).$$

## 3 The Institution $\mathcal{S}$

In this section we introduce the components of the institution $\mathcal{S}$ formalizing the logic underlying the specification language Z. Note that this is not an attempt to give a semantics to the Z specification language. The relationship between $\mathcal{S}$ and Z is similar to the relationship between the institution of equational logic and the semantics of a specification language based on this institution.

### 3.1 Signatures

A signature $\Sigma$ in $\mathrm{SIGN}_{\mathcal{S}}$ consists of a set of names for *given-sets* $G$ and a set of identifiers $O$. Each identifier $id$ in $O$ is associated with a type $\tau(id)$ built from the names of given-sets and the constructors: cartesian product, power-set and schema-type. Note that $\mathcal{S}$ has no type constructors for function types. Instead, a function from $T_1$ to $T_2$ is identified with its graph and is of type $\mathcal{P}(T_1 \times T_2)$. This allows functions to be treated as sets and admits higher-order functions, as functions may take as arguments the graph of a function and also return the graph of a function.

**Definition 2 (Signatures).** *Let $F$ and $V$ be two disjoint recursive enumerable sets of names. A signature $\Sigma$ in $\mathrm{SIGN}_{\mathcal{S}}$ is a tuple $(G, O, \tau)$ where $G$ and $O$ are finite disjoint subsets of $F$. The function $\tau$ maps names in $O$ to types in $\mathcal{T}(G)$ where $\mathcal{T}(G)$ is inductively defined by:*

- $G \subseteq \mathcal{T}(G)$
- *(product)* $T_1 \times \cdots \times T_n \in \mathcal{T}(G)$ *for* $T_i \in \mathcal{T}(G)$, $1 \le i \le n$
- *(power-set)* $\mathcal{P}(T) \in \mathcal{T}(G)$ *for* $T \in \mathcal{T}(G)$
- *(schema-type)* $<x_1 : T_1, \ldots, x_n : T_n> \in \mathcal{T}(G)$ *for* $T_i \in \mathcal{T}(G)$ *and* $x_i \in V$ *and* $x_i \neq x_j$ *for* $1 \le i, j \le n$.

The function $\mathcal{T}$, mapping a given-set $G$ to $\mathcal{T}(G)$, is extended to a functor from $\mathrm{SET}$ to $\mathrm{SET}$ by extending the function $f : G \to G'$ to a function $\mathcal{T}(f) : T(G) \to T(G')$ as follows:

- $\mathcal{T}(f)(g) = g$ for $g \in G$,
- $\mathcal{T}(f)(T_1 \times \ldots \times T_n) = \mathcal{T}(f)(T_1) \times \ldots \times \mathcal{T}(f)(T_n)$ for $T_1, \ldots, T_n \in \mathcal{T}(G)$,
- $\mathcal{T}(f)(\mathcal{P}(T)) = \mathcal{P}(\mathcal{T}(f)(T))$ for $T \in \mathcal{T}(G)$,
- $\mathcal{T}(f)(<x_1 : T_1, \ldots, x_n : T_n>) = <x_1 : \mathcal{T}(f)(T_1), \ldots, x_n : \mathcal{T}(f)(T_n)>$ for $T_1, \ldots, T_n \in \mathcal{T}(G)$.

**Definition 3 (Signature-Morphisms).** *A signature morphism $\sigma$ from a signature $(G, O, \tau)$ to a signature $(G', O', \tau')$ is a pair of functions $\sigma_G : G \to G'$ and $\sigma_O : O \to O'$ such that $\sigma_G$ and $\sigma_O$ are compatible with $\tau$ and $\tau'$, that is $\tau; \mathcal{T}(\sigma_G) = \sigma_O; \tau'$.*

The category $\mathrm{SIGN}_{\mathcal{S}}$ has as objects signatures $\Sigma = (G, O, \tau)$ and as morphisms signature morphisms $\sigma = (\sigma_G, \sigma_O)$ as defined above.

*Example 1.* As an example of a signature in $\mathrm{SIGN}_{\mathcal{S}}$ consider the following small Z specification of a bank account which defines a given set *Integer*, an identifier $+$, and a schema $ACCOUNT$:

$[Integer]$

$+ : Integer \times Integer \rightarrow Integer$

$ACCOUNT$

$bal : Integer$

The signature of this specification is $\Sigma = (\{\mathit{Integer}\}, \{+, ACCOUNT\}, \tau)$ where $\tau$ maps $ACCOUNT$ to the type *Integer* and $+$ to the type $\mathcal{P}(\mathit{Integer} \times \mathit{Integer} \times \mathit{Integer})$. Note that the function type of $+$ is translated to the type $\mathcal{P}(\mathit{Integer} \times \mathit{Integer} \times \mathit{Integer})$ of its graph.

A property necessary for writing modular specifications is the cocompleteness of the category of signatures of an institution.

**Theorem 1.** *The category* $\text{SIGN}_{\mathcal{S}}$ *is finitely cocomplete.*

The colimit of a functor $F : J \rightarrow \text{SIGN}_{\mathcal{S}}$ is given by the colimits of the set of given-set names and the set of identifiers. Note that $\text{SIGN}_{\mathcal{S}}$ is only finitely cocomplete because we have assumed that the set of given-set names and the set of identifiers are finite.

### 3.2 Structures

Given a signature $\Sigma = (G, O, \tau)$ a $\Sigma$-structure $A$ interprets each given-set in $G$ as a set from $\text{SET}$ and each identifier $id$ in $O$ as a value of the set corresponding to the type of $id$.

**Definition 4 ($\Sigma$-structures).** *For a given signature $\Sigma = (G, O, \tau)$ the category $\mathsf{Str}_{\mathcal{S}}(\Sigma)$ of $\Sigma$-structures has as objects pairs $(A_G, A_O)$ where $A_G$ is a functor from the set $G$, viewed as a discrete category, to* $\text{SET}$, *and $A_O$ is the set $\{(o_1, v_1), \dots, (o_n, v_n)\}$ for $O = \{o_1, \dots, o_n\}$ and $v_i \in \bar{A}_G(\tau(o_i))$. The functor $\bar{A}_G : \mathcal{T}(G) \rightarrow \text{SET}$ is given by:*

- $\bar{A}_G(T) = A_G(T)$ *for* $T = g$ *and* $g \in G$
- $\bar{A}_G(T_1 \times \cdots \times T_n) = (\bar{A}_G(T_1) \times \cdots \times \bar{A}_G(T_n))$ *for* $T_1 \times \cdots \times T_n \in \mathcal{T}(G)$
- $\bar{A}_G(\mathcal{P}(T)) = 2^{\bar{A}_G(T)}$ *for* $\mathcal{P}(T) \in \mathcal{T}(G)$
- $\bar{A}_G(<x_1 : T_1, \dots, x_n : T_n>)$
  $$= \{\{(x_1, v_1), \dots, (x_n, v_n)\} \mid v_i \in \bar{A}_g(T_i),\ i \in 1 \dots n\}$$
  *for* $<x_1 : T_1, \dots, x_n : T_n> \in \mathcal{T}(G)$.

*Example 2.* An example of a structure $A$ over the signature defined in Ex. 1 consists of a function $A_G$ mapping *Integer* to $\mathbb{Z}$ and the set

$$A_O = \{(ACCOUNT, \{\{(\mathit{bal}, n)\} \mid n \in \mathbb{Z}\}), (+, \mathit{graph}(\lambda(x, y).x + y))\}.$$

The notation $\mathit{graph}(f)$ is used to denote the graph of a function $f : T \rightarrow T'$.

A morphism $h$ from a $\Sigma$-structure $A$ to a $\Sigma$-structure $B$ is a family of functions between the interpretations of the given-sets which is compatible with the interpretations of the identifiers in $O$.

**Definition 5** (**$\Sigma$-homomorphism**). *A $\Sigma$-homomorphism $h$ from a structure $A = (A_G, A_O)$ to a structure $B = (B_G, B_O)$ is a natural transformation $h : A_G \Rightarrow B_G$ for which $\bar{h}_{\tau(o)}(v_A) = v_B$ for all $o \in O$, $(o, v_A) \in A_O$ and $(o, v_B) \in B_O$ holds. $\bar{h}$ is the extension of $h : A_G \Rightarrow B_G$ to $h : \bar{A}_G \Rightarrow \bar{B}_G$ given by:*

- $\bar{h}_T(v) = h_T(v)$ *for* $T \in G$
- $\bar{h}_T((v_1, \ldots, v_n)) = (\bar{h}_{T_1}(v_1), \ldots, \bar{h}_{T_n}(v_n))$ *for* $T = T_1 \times \cdots \times T_n \in \mathcal{T}(G)$
- $\bar{h}_T(S) = \{\bar{h}_{T'}(v) \mid v \in S\}$ *for* $T = \mathcal{P}(T') \in \mathcal{T}(G)$
- $\bar{h}_T(\{(x_1, v_1), \ldots, (x_n, v_n)\}) = \{(x_1, \bar{h}_{T_1}(v_1)), \ldots, (x_n, \bar{h}_{T_n}(v_n))\}$
  *for* $T = <x_1 : T_1, \ldots, x_n : T_n> \in \mathcal{T}(G)$

**Definition 6** (**$\sigma$-reduct**). *Given a signature morphism $\sigma$ from $\Sigma = (G, O, \tau)$ to $\Sigma' = (G', O', \tau')$ in $\text{SIGN}_{\mathcal{S}}$ and a structure $A = (A_G, A_O)$ in $\mathsf{Str}_{\mathcal{S}}(\Sigma')$ the $\sigma$-reduct of $A$, written $A|_\sigma$, is the structure $B = (B_G, B_O)$ given by:*

- $B_G = \sigma_G; A_G$
- $B_O = \{(o, v) \mid (\sigma_O(o), v) \in A_O,\ o \in O\}$

*For a $\Sigma'$-homomorphism $h : A \to B$ the $\sigma$-reduct is defined as $h|_\sigma = \sigma_G; h$.*

**Definition 7** (**$\mathsf{Str}_{\mathcal{S}}$**). *The contravariant functor $\mathsf{Str}_{\mathcal{S}}$ from $\text{SIGN}_{\mathcal{S}}$ to $\text{CAT}$ assigns to each signature $\Sigma$ the category having as objects $\Sigma$-structures and as morphisms $\Sigma$-homomorphisms, and to each $\text{SIGN}_{\mathcal{S}}$-morphism $\sigma$ from $\Sigma$ to $\Sigma'$ a functor from the category $\mathsf{Str}_{\mathcal{S}}(\Sigma')$ to the category $\mathsf{Str}_{\mathcal{S}}(\Sigma)$ mapping a $\Sigma$-structure $A$ and a $\Sigma$-homomorphism to their $\sigma$-reduct.*

If an institution has amalgamation, two structures $A$ and $B$ over different signatures $\Sigma_A$ and $\Sigma_B$ can be always combined provided that the common components of both signatures are interpreted the same in $A$ and $B$. This allows to build larger structures from smaller ones in a modular way. An institution has amalgamation if and only if its structure functor preserves pushouts, i.e. maps pushout diagrams in $\text{SIGN}_{\mathcal{I}}$ to pullback diagrams in the category of categories. The functor $\mathsf{Str}_{\mathcal{S}}$ not only preserves pushouts but also arbitrary finite colimits.

**Theorem 2.** *The functor $\mathsf{Str}_{\mathcal{S}}$ preserves finite colimits.*

### 3.3 Expressions

The $\Sigma$-formulas are first-order formulas over expressions denoting sets and elements in sets. Expressions can be tested for equality and membership. An important category of expressions, called schema-expressions, denote sets of elements of schema-type.

$$\begin{aligned} E ::= &\ id \mid (E, \ldots, E) \mid E.i \mid <x_1 := E, \ldots, x_n := E> \mid E.x \mid E(E) \\ &\mid \{E, \ldots, E\} \mid \{S \bullet E\} \mid \mathcal{P}(E) \mid E \times \ldots \times E \mid S \end{aligned}$$

The function application $E_1(E_2)$ is well-formed if $E_1$ is of type $\mathcal{P}(T_1 \times T_2)$ and $E_2$ is of type $T_1$. The result is of type $T_2$. If $E_1$ represents the graph of a total function then $E_1(E_2)$ yields the result of that function applied to $E_2$. However, if $E_1$ is the graph of a partial function, or not functional at all, then an arbitrary value from the $\bar{A}_G(T_2)$ is chosen as the result for the situations where $E_2$ is not in the domain of that function or if several results are associated with $E_2$ in $E_1$.

Given a signature $\Sigma = (G, O, \tau)$ and a set of variables $X \subseteq V$ together with a function $\tau_X : X \to \mathcal{T}(G)$ then an *environment* $\epsilon$ is a pair $(\Sigma, (X, \tau_X))$. We use the notation $\epsilon[<x_1 : T_1, \ldots, x_n : T_n>]$ to denote the environment $(\Sigma, (X', \tau'_X))$ given by $X' = X \cup \{x_1, \ldots, x_n\}$ and

$$\tau'_X(id) = \begin{cases} T_i & \text{if } id = x_i \text{ for some } 1 \leq i \leq n \\ \tau_X(id) & \text{else} \end{cases}$$

An expression $E$ is well-formed with respect to $\epsilon$ if

- $E = id$ and $id \in X \cup O \cup G$. The type of $E$ wrt. $\epsilon$ is

$$\tau^\epsilon(E) = \begin{cases} \tau_X(id) & \text{if } id \text{ is in } X, \\ \tau(id) & \text{if } id \text{ is in } O, \\ \mathcal{P}(id) & \text{if } id \text{ is in } G. \end{cases}$$

- $E = (E_1, \ldots, E_n)$ and each $E_i$ is well-formed for all $1 \leq i \leq n$. Then $\tau^\epsilon(E) = \tau^\epsilon(E_1) \times \ldots \times \tau^\epsilon(E_n)$.
- $E = E'.i$, $\tau^\epsilon(E') = T_1 \times \ldots \times T_n$ and $1 \leq i \leq n$. The type of $E$ is $T_i$.
- $E = <x_1 := E_1, \ldots, x_n := E_n>$, $x_i \in V$, $x_i \neq x_j$ and each $E_i$ is well-formed. The type of $E$ is $<x_1 : \tau^\epsilon(E_1), \ldots, x_n : \tau^\epsilon(E_n)>$.
- $E = E'.x$, $\tau^\epsilon(E') = <x_1 : T_1, \ldots, x_n : T_n>$ and $x = x_i$ for some $1 \leq i \leq n$. The type of $E$ is $T_i$.
- $E_1(E_2)$, $\tau^\epsilon(E_1) = \mathcal{P}(T_1 \times T_2)$ and $\tau^\epsilon(E_2) = T_1$. The type of $E$ is $T_2$.
- $E = \{E_1, \ldots, E_n\}$, each $E_i$ is well-formed and all $E_i$ have the same type $T$ for $1 \leq i \leq n$. The type of $E$ is $\mathcal{P}(T)$.
- $E = \{S \bullet E'\}$, $S$ is well-formed and has type $\mathcal{P}(<x_1 : T_1, \ldots, x_n : T_n>)$ and $E'$ is well-formed with respect to $\epsilon[<x_1 : T_1, \ldots, x_n : T_n>]$. The type of $E$ is $\mathcal{P}(\tau^{\epsilon'}(E'))$.
- $E = \mathcal{P}(E')$ and $E'$ is well-formed. The type of $E$ is $\mathcal{P}(\tau^\epsilon(E'))$.
- $E = E_1 \times \ldots \times E_n$ and each $E_i$ is well-formed. The type of $E$ is $\mathcal{P}(\tau^\epsilon(E_i) \times \ldots \times \tau^\epsilon(E_n))$.
- $E = S$ and $S$ is a well-formed schema-expression with respect to $\epsilon$ (well-formedness of schema-expressions is defined later in this paper.) The type of $E$ is the type of $S$ with respect to $\epsilon$.

Let $E$ be an expression well-formed with respect to an environment $\epsilon = (\Sigma, (X, \tau_X))$ and let $A = (A_G, A_O)$ be a $\Sigma$-structure. The semantics of an expression $E$ is given with respect to a *variable binding* $\beta$ compatible with the environment $\epsilon$. A variable binding $\beta = (A, A_X)$ *compatible* with $\epsilon$ consists of a $\Sigma$-structure $A$ and a set $A_X = \{(x_1, v_1) \ldots (x_n, v_n)\}$ with $v_i \in \bar{A}_G(\tau_X(x_i))$ for all $1 \leq i \leq n$.

If $v = \{(x_1, v_1), \ldots, (x_n, v_n)\}$ is an element of type $T = <x_1 : T_1, \ldots, x_n : T_n>$ then the notation $\beta[v]$ is used to describe the variable binding $(A, A'_X)$ where $(x_i, v_i)$ is in $A'_x$ iff $(x_i, v_i)$ is in $v$, or there is no $(x_i, v'_i)$ in $v$ for some $v_i$ and $(x_i, v_i)$ is in $A_X$.

Now the semantics of an expression $E$ wrt. $\beta$ is defined as follows:

- $[\![id]\!]^\beta = v$ if $(id, v) \in A_X$ and $id \in X$ or $(id, v) \in A_O$ and $o \in O$, or $[\![id]\!]^\beta = A_G(id)$ if $id$ is in $G$.
- $[\![(E_1, \ldots, E_n)]\!]^\beta = ([\![E_1]\!]^\beta, \ldots, [\![E_n]\!]^\beta)$.
- $[\![E.i]\!]^\beta = v_i$ if $[\![E]\!]^\beta = (v_1, \ldots, v_n)$.
- $[\![<x_1 := E_1, \ldots, x_n := E_n>]\!]^\beta = \{(x_1, [\![E_1]\!]^\beta), \ldots, (x_n, [\![E_n]\!]^\beta)\}$.
- $[\![E.x]\!]^\beta = v_i$ if $[\![E]\!]^\beta = \{(x_1, v_1), \ldots, (x_n, v_n)\}$ and $x = x_i$.
- $[\![E_1(E_2)]\!]^\beta = v$ if $v$ is unique with $([\![E_2]\!]^\beta, v)$ in $[\![E_1]\!]^\beta$. If another $v'$ with $([\![E_2]\!]^\beta, v')$ in $[\![E_1]\!]^\beta$ exists or if none exists then $v$ is an arbitrary element of $A_G(T_2)$, where $\tau^\epsilon(E_1) = \mathcal{P}(T_1 \times T_2)$.
- $[\![\{E_1, \ldots, E_n\}]\!]^\beta = \{[\![E_1]\!]^\beta, \ldots, [\![E_n]\!]^\beta\}$.
- $[\![\{S \bullet E\}]\!]^\beta = \{[\![E]\!]^{\beta[v]} \mid v \in [\![S]\!]^\beta\}$.
- $[\![\mathcal{P}(E)]\!]^\beta = 2^{[\![E]\!]^\beta}$.
- $[\![E_1 \times \ldots \times E_n]\!]^\beta = [\![E_1]\!]^\beta \times \ldots \times [\![E_n]\!]^\beta$.

**Schema-expressions** A schema denotes a set of elements of schema-type, which have the form $\{(x_1, v_1), \ldots, (x_n, v_n)\}$ and are called *bindings*. Thus the *type of a schema* is $\mathcal{P}(<x_1 : T_1, \ldots, x_n : T_n>)$ if $T_i$ is the type of $v_i$ for $1 \leq i \leq n$. A simple schema of the form $x_1 : E_1, \ldots, x_n : E_n$ defines the identifiers of a schema and a set of possible values for each identifier. Given a schema $S$ we can a define a new schema $S|P$ having as elements all the elements of $S$ satisfying the predicate $P$. We can form the negation, disjunction, conjunction and implication of schema-expressions, which correspond to the complement, union and intersection of the sets denoted by the arguments. For the disjunction, conjunction and implication of schema-expressions the type of the arguments have to be compatible, that is, if two components have the same name, they have to have the same type. The type of the result has as components the union of the components of the arguments with all duplicates removed. Adjustments of the type of schemas can be made by using hiding and renaming, where hiding hides some components of a schema-type and renaming renames some components. A particular kind of renaming is decorating the identifiers with finite sequences of elements from $\{', !, ?\}$. An existentially quantified schema $\exists S_1 . S_2$ denotes the set of all bindings of the identifiers of $S_2$ without the ones in $S_1$ such that there exists a binding in $S_1$ such that the union of the bindings is an element of $S_2$. An universally quantified schema $\forall S_1 . S_2$ is an abbreviation for $\neg \exists S_1 . \neg S_2$.

$$\begin{aligned} S ::= {} & x_1 : E, \ldots, x_n : E \mid (S|P) \mid \neg S \mid S \vee S \mid S \wedge S \mid S \Rightarrow S \\ & \mid \forall S.S \mid \exists S.S \mid S \setminus [x_1, \ldots, x_n] \mid S[x_1/y_1, \ldots, x_n/y_n] \\ & \mid S\ Decor \mid E \end{aligned}$$

Note that the schema operations $\Delta S$ and $\Theta S$, used in Z for the specification of sequential systems, are only convenient abbreviations for schema expressions involving the schema operations defined above. For example, $\Delta S$ is the same as the conjunction of the schema $S$ with $S'$, where $S'$ is $S$ where all components are decorated with a prime, and $\Theta S$ is the same as the schema $\Delta S | (x_1 = x_1' \wedge \ldots \wedge x_n = x_n')$ if $\mathcal{P}(<x_1 : T_1, \ldots, x_n : T_n>)$ is the type of $S$.

A schema-expression $S$ is well-formed with respect to an environment $\epsilon = (\Sigma, (X, \tau_X))$ with $\Sigma = (G, O, \tau)$, if

- $S = x_1 : E_1, \ldots, x_n : E_n$, $x_i \in V$ and $E_i$ is well-formed and has type $\mathcal{P}(T_i)$ for each $1 \leq i \leq n$. The type of $S$ is $\mathcal{P}(<x_1 : T_1, \ldots, x_n : T_n>)$.
- $S = S'|P$ and $P$ is well-formed with respect to $\epsilon' = \epsilon[T]$, where $\mathcal{P}(T)$ is the type of $S'$ with respect to $\epsilon$. The type of $S$ is $\mathcal{P}(T)$.
- $S = \neg S'$ and $S'$ is well-formed. The type of $S$ is $\tau^{\epsilon}(S')$.
- $S = S_1 \ op \ S_2$, $S_1$ and $S_2$ have compatible types, and $S_1$ and $S_2$ are well-formed for each $op \in \{\vee, \wedge, \Rightarrow\}$. Two types $\mathcal{P}(<x_1 : T_1, \ldots, x_n : T_n>)$ and $\mathcal{P}(<x'_1 : T'_1, \ldots, x'_m : T'_m>)$ are compatible if for all $i, j$ such that $x_i = x'_j$ we have $T_i = T'_j$. The type of $S$ has as components the union of the components of the type of $S_1$ and $S_2$ with the duplicates removed.
- $S = \exists S_1.S_2$, $S_1$ and $S_2$ are well-formed with respect to $\epsilon$ and their types are compatible. The type of $S$ is the type of $S_2$ with all the identifiers removed which occur in $S_1$.
- $S = S' \setminus [x_1, \ldots, x_n]$ and $S$ is well-formed. Note that it is not required that the $x_i$ have to be identifiers of the type of $S'$. The type of $S$ is the type of $S'$ without the identifier $x_i$ if $x_i$ occurs in the type of $S$ for all $1 \leq i \leq n$.
- $S = S'[x_1/y_1, \ldots, x_n/y_n]$ and $S$ is well-formed. Note that it is not required that the $x_i$ have to be identifiers of the type of $S'$. The type of $S$ is the type of $S'$ where $x_i$ is replaced by $y_i$ if $x_i$ is an identifier of $S'$. Note that the function from the identifiers of the type of $S'$ to the identifiers of the type of $S$ defined by this replacement has to be injective.
- $S = S'$ *Decor* and $S'$ is well-formed. *Decor* is a finite sequence of elements from $\{', !, ?\}$. The type of $S$ is $\mathcal{P}(<\bar{x}_1 : T_1, \ldots, \bar{x}_n : T_n>)$ if $S'$ is of type $\mathcal{P}(<x_1 : T_1, \ldots, x_n : T_n>)$. $\bar{x}_i$ is the decorated form of $x_i$, for example, if *Decor* is ! then $\bar{x}_i$ is $x_i!$.
- $S = E$ and $E$ is well-formed with type $\mathcal{P}(<x_1 : T_1, \ldots, x_n : T_n>)$. The type of $S$ is $\mathcal{P}(<x_1 : T_1, \ldots, x_n : T_n>)$.

Let $v$ be the set $\{(x_1, v_1), \ldots, (x_n, v_n)\}$ and $X$ be a set of variables, then $v|_X$ denotes the binding $v$ restricted to the identifiers in the set $X$, i.e. the set $\{(x_i, v_i) \mid x_i \in X, \ (x_i, v_i) \in v\}$.

If a schema-expression $S$ is well-formed with respect to $\epsilon$, its semantics $[\![S]\!]^{\beta}$ with respect to a structure $A = (A_G, A_O)$ and a variable binding $\beta = (A, A_X)$ compatible with $\epsilon$ is defined as follows:

- $[\![x_1 : E_1, \ldots, x_n : E_n]\!]^{\beta} = \{\{(x_1, v_1), \ldots, (x_n, v_n)\} \mid v_i \in [\![E_i]\!]^{\beta}, \ 1 \leq i \leq n\}$.
- $[\![S|P]\!]^{\beta} = \{v \in [\![S]\!]^{\beta} \mid \beta[v] \models^{\mathcal{S}} P\}$. The satisfaction relation $\models^{\mathcal{S}}$ is defined in Sect. 3.4.
- $[\![\neg S]\!]^{\beta} = \{v \in \bar{A}_G(T) \mid v \notin [\![S]\!]^{\beta}\}$ and $T$ is the type of $S$.
- $[\![S \setminus [y_1, \ldots, y_n]]\!]^{\beta} = \{v|_{\{x_1, \ldots, x_m\}} \mid v \in [\![S]\!]^{\beta}\}$, where $\{x_1, \ldots, x_m\}$ is the set of identifiers of the type of $S$ without the identifiers $y_1, \ldots, y_n$.
- $[\![S_1 \ op \ S_2]\!]^{\beta} = \{v \in \bar{A}_G(T) \mid v|_{X_1} \in [\![S_1]\!]^{\beta} \ op \ v|_{X_2} \in [\![S_2]\!]^{\beta}\}$ for $op \in \{\vee, \wedge, \Rightarrow\}$, where $T$ is the type of $S_1 \ op \ S_2$ and $X_1$ and $X_2$ are the set of components of schemata $S_1$ and $S_2$, respectively. Note that $v \in \bar{A}_G(T)$ guarantees that if $(x, a) \in v_1$, $(x, a') \in v_2$ and $v = v_1 \cup v_2$ then $a = a'$.
- $[\![\exists S_1.S_2]\!]^{\beta} = \{v \in \bar{A}_G(\tau^{\epsilon}(\exists S_1.S_2)) \mid \exists v_1 \in [\![S_1]\!]^{\beta} \ (v_1 \cup v)|_{X_2} \in [\![S_2]\!]^{\beta}\}$ where $X_2$ is the set of components of schema $S_2$.
- $[\![S[y_1/y'_1, \ldots, y_n/y'_n]]\!]^{\beta} = \{\bar{f}(v) \mid v \in [\![S]\!]^{\beta}\}$ where $f$ is the function from the identifiers of type $S$ to the identifiers of type $S'$ defined by $[y_1/y'_1, \ldots, y_n/y'_n]$

as follows:

$$f(id) = \begin{cases} y_i' & \text{if } y_i = id \text{ for some } 1 \leq i \leq n \\ id & \text{else} \end{cases}$$

and $\bar{f}$ is the extension of $f$ to bindings.

- $[\![S'\ Decor]\!]^\beta = \{\{(\bar{x}_1, v_1), \ldots, (\bar{x}_n, v_n)\} \mid \{(x_1, v_1), \ldots, (x_n, v_n)\} \in [\![S']\!]^\beta\}$. $\bar{x}_i$ is the identifier $x_i$ decorated with *Decor*. For example, if *Decor* is $'$ then $\bar{x}_i$ is $x_i{}'$.

### 3.4 Formulas

The formulas in $\mathsf{Sen}_{\mathcal{S}}(\Sigma)$ are the usual first-order formulas built on the membership predicate and the equality between expressions.

$$\begin{aligned} P ::= &\ \text{true} \mid \text{false} \mid E \in E \mid E = E \mid \neg P \mid P \vee P \mid P \wedge P \\ &\mid P \Rightarrow P \mid \forall S.P \mid \exists S.P \end{aligned}$$

A formula $P$ is well-formed in an environment $\epsilon = (\Sigma, (X, \tau_X))$ if

- $P = E_1 \in E_2$, $\tau^\epsilon(E_2) = \mathcal{P}(\tau^\epsilon(E_1))$ and $E_1$ and $E_2$ are well-formed.
- $P = (E_1 = E_2)$, $\tau^\epsilon(E_1) = \tau^\epsilon(E_2)$ and $E_1$ and $E_2$ are well-formed.
- $P = \neg P'$ and $P'$ is well-formed.
- $P = P_1\ op\ P_2$ and $P_1$ and $P_2$ are well-formed for $op \in \{\vee, \wedge, \Rightarrow\}$.
- $P = \forall S.P'$, $S$ is well-formed and has type $\mathcal{P}(T)$ where $T$ is a schema-type and $P'$ is well-formed with respect to $\epsilon[T]$.
- $P = \exists S.P'$, $S$ is well-formed and has type $\mathcal{P}(T)$ where $T$ is a schema-type and $P'$ is well-formed with respect to $\epsilon[T]$.

Given a signature-morphism $\sigma : \Sigma \to \Sigma'$ and a formula $P$ well-formed with respect to $\epsilon = (\Sigma, (X, \tau_X))$ then the formula $\bar{\sigma}(P)$ is well-formed with respect to $(\Sigma', (X, \tau_X'))$ where $\tau_X' = \tau_X; T(\sigma_G)$ and $\bar{\sigma}(P)$ is given by:

- $\bar{\sigma}(id) = id$ if $id \in X$, $\bar{\sigma}(id) = \sigma_O(id)$ if $id \in O$ and $\bar{\sigma}(id) = \sigma_G(id)$ if $id \in G$.
- $\bar{\sigma}((E_1, \ldots, E_n)) = (\bar{\sigma}(E_1), \ldots, \bar{\sigma}(E_n))$.
- $\bar{\sigma}(E.i) = \bar{\sigma}(E).i$.
- $\bar{\sigma}(<x_1 := E_1, \ldots, x_n := E_n>) = <x_1 := \bar{\sigma}(E_1), \ldots, x_n := \bar{\sigma}(E_n)>$.
- $\bar{\sigma}(E.x) = \bar{\sigma}(E).x$.
- $\bar{\sigma}(E_1(E_2)) = \bar{\sigma}(E_1)(\bar{\sigma}(E_2))$.
- $\bar{\sigma}(\{E_1, \ldots, E_n\}) = \{\bar{\sigma}(E_1), \ldots, \bar{\sigma}(E_n)\}$.
- $\bar{\sigma}(\{S \bullet E\}) = \{\bar{\sigma}(S) \bullet \bar{\sigma}(E)\}$.
- $\bar{\sigma}(\mathcal{P}(E)) = \mathcal{P}(\bar{\sigma}(E))$.
- $\bar{\sigma}(E_1 \times \ldots \times E_n) = \bar{\sigma}(E_1) \times \ldots \times \bar{\sigma}(E_n)$.
- $\bar{\sigma}(x_1 : E_1, \ldots, x_n : E) = x_1 : \bar{\sigma}(E_1), \ldots, x_n : \bar{\sigma}(E_n)$.
- $\bar{\sigma}(S|P) = \bar{\sigma}(S)|\bar{\sigma}(P)$.
- $\bar{\sigma}(\neg S) = \neg\bar{\sigma}(S)$.
- $\bar{\sigma}(S_1\ op\ S_n) = \bar{\sigma}(S_1)\ op\ \bar{\sigma}(S_n)$ for $op \in \{\vee, \wedge, \Rightarrow\}$.
- $\bar{\sigma}(\exists S_1.S_2) = \exists\bar{\sigma}(S_1).\bar{\sigma}(S_2)$ and $\bar{\sigma}(\forall S_1.S_2) = \forall\bar{\sigma}(S_1).\bar{\sigma}(S_2)$.
- $\bar{\sigma}(S \setminus [x_1, \ldots, x_n]) = \bar{\sigma}(S) \setminus [x_1, \ldots, x_n]$.
- $\bar{\sigma}(S[x_1/y_1, \ldots, x_n/y_n]) = \bar{\sigma}(S)[x_1/y_1, \ldots, x_n/y_n]$.
- $\bar{\sigma}(E_1 \in E_2) = \bar{\sigma}(E_1) \in \bar{\sigma}(E_2)$.

- $\bar{\sigma}(E_1 = E_2) = \bar{\sigma}(E_1) = \bar{\sigma}(E_n)$.
- $\bar{\sigma}(\text{true}) = \text{true}$ and $\bar{\sigma}(\text{false}) = \text{false}$.
- $\bar{\sigma}(\neg P) = \neg\bar{\sigma}(P)$.
- $\bar{\sigma}(P_1 \ op\ P_2) = \bar{\sigma}(P_1)\ op\ \bar{\sigma}(P_2)$ for $op \in \{\vee, \wedge, \Rightarrow\}$.
- $\bar{\sigma}(\forall S.P) = \forall\bar{\sigma}(S).\bar{\sigma}(P)$ and $\bar{\sigma}(\exists S.P) = \exists\bar{\sigma}(S).\bar{\sigma}(P)$.

**Definition 8 ($\mathsf{Sen}_\mathcal{S}$).** *The functor $\mathsf{Sen}_\mathcal{S}$ from $\mathrm{SIGN}_\mathcal{S}$ to $\mathrm{SET}$ maps each signature $\Sigma$ to its set of $\Sigma$-formulas and each signature morphism $\sigma$ from $\Sigma$ to $\Sigma'$ to the translation of $\Sigma$-formulas to $\Sigma'$-formulas given by $\bar{\sigma}$.*

Validity of a well-formed formula $P$ in $\beta = (A, A_X)$, $\beta \models^\mathcal{S} P$, is defined by:

- $\beta \models^\mathcal{S}$ true.
- $\beta \models^\mathcal{S} E_1 \in E_2$ iff $[\![E_1]\!]^\beta \in [\![E_2]\!]^\beta$.
- $\beta \models^\mathcal{S} E_1 = E_2$ iff $[\![E_1]\!]^\beta = [\![E_2]\!]^\beta$.
- $\beta \models^\mathcal{S} \neg P$ iff not $\beta \models^\mathcal{S} P$.
- $\beta \models^\mathcal{S} P_1\ op\ P_2$ iff $\beta \models^\mathcal{S} P_1\ op\ \beta \models^\mathcal{S} P_2$ for $op \in \{\vee, \wedge, \Rightarrow\}$.
- $\beta \models^\mathcal{S} \forall S.P$ iff $\beta[v] \models^\mathcal{S} P$ for all $v \in [\![S]\!]^\beta$.
- $\beta \models^\mathcal{S} \exists S.P$ iff $\beta[v] \models^\mathcal{S} P$ for some $v \in [\![S]\!]^\beta$.

**Definition 9 (Satisfaction).** *Given a signature $\Sigma$, a formula $P$ which is well-formed with respect to $(\Sigma, (\{\}, \tau_X))$, and a $\Sigma$-structure $A$ then $A \models^\mathcal{S}_\Sigma P$ if $(A, \{\}) \models^\mathcal{S} P$.*

**Theorem 3 (The Institution $\mathcal{S}$).** *The category $\mathrm{SIGN}_\mathcal{S}$, the functor $\mathsf{Str}_\mathcal{S}$, the functor $\mathsf{Sen}_\mathcal{S}$ and the family of satisfaction relations given by $\models^\mathcal{S}_\Sigma$ define the institution $\mathcal{S} = \langle \mathrm{SIGN}_\mathcal{S}, \mathsf{Str}_\mathcal{S}, \mathsf{Sen}_\mathcal{S}, \models^\mathcal{S} \rangle$.*

*Example 3.* To complete our small example of a bank account we define the schema $\Delta ACCOUNT$ and the operation $UPDATE$ adding $n$ to the balance of the account:

$$\Delta ACCOUNT = ACCOUNT \wedge ACCOUNT'$$

| $UPDATE$ |
|---|
| $\Delta ACCOUNT$<br>$n : Integer$ |
| $bal' = bal + n$ |

The abstract datatype in $\mathcal{S}$ corresponding to this specification consists of the signature:

$$\Sigma_{BA} = (\{Integer\}, \{+, ACCOUNT, \Delta ACCOUNT, UPDATE\}, \tau)$$

where $\tau$ is given by

$$\tau(id) = \begin{cases} \mathcal{P}(Integer \times Integer \times Integer) & \mathit{if}\ id = + \\ \mathcal{P}(<bal : Integer>) & \mathit{if}\ id = ACCOUNT \\ \mathcal{P}(<bal : Integer,\ bal' : Integer>) & \mathit{if}\ id = \Delta ACCOUNT \\ \mathcal{P}(<bal : Integer,\ bal' : Integer,\ n : Integer>) & \mathit{if}\ id = UPDATE \end{cases}$$

The following set of formulas specifies the schema $\Delta ACCOUNT$ and the $UPDATE$ operation:

$$\Phi = \begin{array}{l}\{\Delta ACCOUNT = ACCOUNT \wedge ACCOUNT',\\ UPDATE = ((\Delta ACCOUNT \wedge (n : \mathit{Integer})) \mid \mathit{bal}' = \mathit{bal} + n)\}\end{array}$$

## 4 Relating Abstract Datatypes to Schemata

Let $\Sigma = (G, O, \tau)$ be a signature in $\mathcal{S}$. A schema-type

$$T = <x_1 : T_1, \ldots, x_n : T_n>$$

defines a signature $\Sigma' = (G, O \cup \{x_1, \ldots, x_n\}, \tau')$ where $\tau'(x_i) = T_i$ and $\tau'(id) = \tau(id)$ for $id \in O$.[1]

Given a $\Sigma$-structure $A = (A_G, A_O)$ then an element $\{(x_1, v_1), \ldots, (x_n, v_n)\}$ of type $T$ defines a $\Sigma'$-structure $A' = (A_G, A_O \cup \{(x_1, v_1), \ldots, (x_n, v_n)\})$.

**Definition 10.** *Given a signature $\Sigma = (G, O, \tau)$, a schema-expression $S$ of type $\mathcal{P}(<x_1 : T_1, \ldots, x_n : T_n>)$ and a $\Sigma$-structure $A = (A_G, A_O)$. Define an abstract datatype $(\Sigma_S, M_S^A)$ by*

- $\Sigma_S = (G, O \cup \{x_1, \ldots, x_n\}, \tau_S)$, *where* $\tau_S(x_i) = T_i$ *for* $1 \leq i \leq n$ *and* $\tau_S(id) = \tau(id)$ *for* $id \in O$ *and*
- $M_S^A = \{(A_G, A_O \cup v_S) \mid v_S \in [\![S]\!]^{((A_G, A_O), \{\})}\}$.

*This definition can be extended to abstract datatypes* $\mathrm{SP} = (\Sigma, M)$ *in* $\mathrm{ADT}_{\mathcal{S}}$ *by taking the union of all* $M_S^A$ *for* $A \in M$*:*

$$\mathrm{SP}_S = (\Sigma_S, \bigcup_{A \in M} M_S^A).$$

*Example 4.* Given $\Sigma = (\{\mathit{Integer}\}, \{+\}, \tau)$ then the signatures corresponding to the schemata $ACCOUNT$, $\Delta ACCOUNT$ and $UPDATE$ are:

$$\begin{aligned}\Sigma_A &= (\{\mathit{Integer}\}, \{+, \mathit{bal}\}, \tau_A),\\ \Sigma_{\Delta A} &= (\{\mathit{Integer}\}, \{+, \mathit{bal}, \mathit{bal}'\}, \tau_{\Delta A}),\\ \Sigma_U &= (\{\mathit{Integer}\}, \{+, \mathit{bal}, \mathit{bal}', n\}, \tau_U).\end{aligned}$$

The next theorem relates the operations on schemata with the operations on abstract datatypes:

**Theorem 4.** *Let* $\mathrm{SP} = (\Sigma, M)$ *be an abstract datatype in* $\mathcal{S}$*. If*

[1] Note that $\Sigma'$ is not a signature as defined in Def. 2 because $\{x_1, \ldots, x_n\}$ is not a subset of $F$ since, for technical reasons, we had to require that the set of variable names and the set of identifier names are disjoint. However, we can assume that $O'$ is the set $O \cup \{\bar{x}_1, \ldots, \bar{x}_n\}$ where the $\bar{x}_i$ are suitable renamings of $x_i$ to symbols in $F$ not occurring in $O$.

- $S = x : E_1, \ldots, x : E_n$ *then* $\mathrm{S\scriptstyle P}_S = I_{\{x_i \in E_i | 1 \le i \le n\}} T_\sigma \mathrm{S\scriptstyle P}$ *where* $\sigma$ *is the inclusion of* $\Sigma$ *into* $\Sigma_S$.
- $S = S'|P$ *then* $\mathrm{S\scriptstyle P}_S = I_{\{P\}} \mathrm{S\scriptstyle P}_{S'}$.
- $S = S_1 \wedge S_2$ *then* $\mathrm{S\scriptstyle P}_S = T_{\sigma_1} \mathrm{S\scriptstyle P}_{S_1} + T_{\sigma_2} \mathrm{S\scriptstyle P}_{S_2}$. *The signature morphisms* $\sigma_1$ *and* $\sigma_2$ *are the inclusions of the signatures* $\Sigma_{S_1}$ *and* $\Sigma_{S_2}$ *into* $\Sigma_{S_1 \wedge S_2}$. *This is needed because, in contrast to the union of abstract datatypes, the types of* $S_1$ *and* $S_2$ *are only required to be compatible in the union of* $S_1$ *and* $S_2$.
- $S = S' \setminus [x_1, .., x_n]$ *then* $\mathrm{S\scriptstyle P}_S = D_\sigma \mathrm{S\scriptstyle P}_{S'}$ *where* $\sigma$ *is the inclusion of* $\Sigma_S$ *into* $\Sigma_{S'}$.
- $S = S'[x_1/y_1, .., x_n/y_n]$ *then* $\mathrm{S\scriptstyle P}_S = T_\sigma \mathrm{S\scriptstyle P}_{S'}$ *where* $\sigma_G$ *is the identity and* $\sigma_O(x) = y_i$, *if* $x = x_i$ *for some* $i$ *and* $\sigma_O(x) = x$ *if* $x \neq x_i$ *for all* $i$.

*Example 5.* Given $\mathrm{S\scriptstyle P} = (\Sigma, M)$ and $\mathit{UPDATE} = (\Delta \mathit{ACCOUNT} \wedge (n : \mathit{Integer}) \mid \mathit{bal}' = \mathit{bal} + n)$ we can write $\mathrm{S\scriptstyle P}_U = (\Sigma_U, M_U)$ as:

$$\mathrm{S\scriptstyle P}_U = I_{\{\mathit{bal}' = \mathit{bal} + n\}} (T_{\sigma_1} \mathrm{S\scriptstyle P}_{\Delta A} + T_{\sigma_2} I_{\{n \in \mathit{Integer}\}} T_{\sigma_3} \mathrm{S\scriptstyle P}).$$

Here, $\sigma_1$ is the inclusion of $\Sigma_{\Delta A}$ into $\Sigma_U$, $\sigma_3$ the inclusion of $\Sigma$ into $\Sigma_{(n:\mathit{Integer})}$, and $\sigma_2$ the inclusion of $\Sigma_{(n:\mathit{Integer})}$ into $\Sigma_U$. $\Sigma_{(n:\mathit{Integer})} = (\{\mathit{Integer}\}, \{+, n\}, \tau')$ is the signature corresponding to the schema $(n : \mathit{Integer})$.

What about the other schema operations $\neg S$, $S_1 \vee S_2$, $S_1 \Rightarrow S_2$, and $\exists S_1 . S_2$? The existential quantifier is the same as hiding the schema variables of $S_1$ in the conjunction of $S_1$ and $S_2$. Let $x_1, \ldots, x_n$ be the schema variables of $S_1$ then $\exists S_1 . S_2$ and $(S_1 \wedge S_2) \setminus [x_1, .., x_n]$ have the same semantics. This yields the following theorem:

**Theorem 5.** *Let* $\mathrm{S\scriptstyle P} = (\Sigma, M)$ *be an abstract datatype in* $\mathcal{S}$, *and* $S = \exists S_1 . S_2$ *a well-formed schema expression wrt. the environment* $\epsilon$. *Then*

$$\mathrm{S\scriptstyle P}_S = D_\sigma (T_{\sigma_1} \mathrm{S\scriptstyle P}_{S_1} \wedge T_{\sigma_2} \mathrm{S\scriptstyle P}_{S_2})$$

*where* $\sigma_1$ *and* $\sigma_2$ *are the inclusions of* $\Sigma_{S_1}$ *and* $\Sigma_{S_2}$ *into* $\Sigma_{S_1 \wedge S_2}$, *and* $\sigma$ *is the inclusion of the signature of the whole expression into* $\Sigma_{S_1 \wedge S_2}$.

It is easy to define negation, disjunction and implication on abstract datatypes:

**Definition 11.** *Let* $(\Sigma, M)$, $(\Sigma, M_1)$ *and* $(\Sigma, M_2)$ *be abstract datatypes in an arbitrary institution* $\mathcal{I}$, *define:*

$$\begin{aligned} \neg(\Sigma, M) &= (\Sigma, \{m \in \mathsf{Str}_\mathcal{I}(\Sigma) \mid m \notin M\}) \\ (\Sigma, M_1) \vee (\Sigma, M_2) &= (\Sigma, M_1 \cup M_2) \\ (\Sigma, M_1) \Rightarrow (\Sigma, M_2) &= (\Sigma, \{m \in \mathsf{Str}_\mathcal{I}(\Sigma) \mid m \in M_1 \Rightarrow m \in M_2\}) \end{aligned}$$

What is the relationship of these operations to the corresponding schema operations? Disjunction can be treated similar to conjunction; however, while it seems natural to expect $\mathrm{S\scriptstyle P}_{\neg S} = \neg \mathrm{S\scriptstyle P}_S$, this does not hold. The reason is that in $\mathrm{S\scriptstyle P}_{\neg S}$ the negation of $S$ is interpreted within a given abstract datatype $\mathrm{S\scriptstyle P}$ while the negation of $\mathrm{S\scriptstyle P}_S$ also permits the negation of $\mathrm{S\scriptstyle P}$ itself. If $(A_G, A_O \cup v)$ is a model of $\mathrm{S\scriptstyle P}_{\neg S}$ then $v$ is not in $[\![S]\!]^\beta$ and $(A_G, A_O)$ is always a model of $\mathrm{S\scriptstyle P}$. On the other hand, if $(A_G, A_O \cup v)$ is a model of $\neg \mathrm{S\scriptstyle P}_S$, either $v$ is not in $[\![S]\!]^\beta$ or $(A_G, A_O)$ is not a model of $\mathrm{S\scriptstyle P}$. The solution is to add the requirement that $(A_G, A_O)$ is a model of $\mathrm{S\scriptstyle P}$ to $\neg \mathrm{S\scriptstyle P}_S$. Implication has a similar problem.

**Theorem 6.** *Let* $\mathrm{Sp} = (\Sigma, M)$ *be an abstract datatype in* $\mathcal{S}$*. If*

- $S = S_1 \vee S_2$ *then* $\mathrm{Sp}_S = T_{\sigma_1}\mathrm{Sp}_{S_1} \vee T_{\sigma_2}\mathrm{Sp}_{S_2}$*. The signature morphisms* $\sigma_1$ *and* $\sigma_2$ *are the inclusions of the signatures* $\Sigma_{S_1}$ *and* $\Sigma_{S_2}$ *into* $\Sigma_{S_1 \vee S_2}$*.*
- $\mathrm{Sp}_{\neg S} = \neg \mathrm{Sp}_S + T_{\sigma_S}\mathrm{Sp}$ *where* $\sigma_S$ *is the inclusion of the* $\Sigma$ *into* $\Sigma_S$*.*
- $S = S_1 \Rightarrow S_2$ *then* $\mathrm{Sp}_S = (T_{\sigma_1}\mathrm{Sp}_{S_1} \Rightarrow T_{\sigma_2}\mathrm{Sp}_{S_2}) + T_{\sigma_S}\mathrm{Sp}$*. The signature morphisms* $\sigma_1$ *and* $\sigma_2$ *are the inclusions of the signatures* $\Sigma_{S_1}$ *and* $\Sigma_{S_2}$ *into* $\Sigma_{S_1 \Rightarrow S_2}$*.*

## 5 Conclusion

In this paper we have formalized the relationship between the structuring mechanism in Z and the structuring mechanism of property-oriented specification languages. Z specifications are structured using schemata and operations on schemata, which are based on the particular logic underlying Z. In contrast, property-oriented specifications are structured using abstract datatypes and operations on abstract datatypes, which can be formulated largely independent of the logic used for the specifications.

The advantage of having the structuring mechanism represented as part of the logic is that it is possible to reason within that logic about the structure of specifications. The disadvantage is that it is not easy to transfer results and methods to be used with a different logic and specification language. For example, the specification of sequential systems in Z consists of a schema for the state space and a schema for each operation. In the example of the bank account the schema *ACCOUNT* defines the state space of the bank account and the schema *UPDATE* defines the update operation that changes the state of the account. Using the results of this paper we can use abstract datatypes instead of schemata for the specification of sequential systems and the bank account specification can be written without the use of schemata as a CASL-$\mathcal{S}$ specification as follows:

**spec** *BASE* =
  sort Integer
  **op** $+ : \mathcal{P}(\mathit{Integer} \times \mathit{Integer} \times \mathit{Integer})$

**spec** *ACCOUNT* = *BASE* **then**
  **op** bal : Integer

**spec** $\Delta$*ACCOUNT* = *ACCOUNT* **and** { *ACCOUNT* **with** $bal \mapsto bal'$ }

**spec** *UPDATE* = $\Delta$*ACCOUNT* **then**
  **op** $n : \mathit{Integer}$
  **axioms** $bal' = bal + n$

Note that this specification does not make any reference to schemata anymore. Instead of schemata the structuring facilities of CASL-$\mathcal{S}$ are used. Since these structuring facilities, based on abstract datatypes and operations on abstract datatypes, are institution independent[2], this allows the use of the Z-style for the specification of sequential systems also with other specification languages.

[2] To be precise, CASL is parameterized by the notion of an institution with symbols (cf. Mossakowski [8]). However, it is easy to show that $\mathcal{S}$ is an institution with symbols.

For example, this specification style can be used in the state as algebra approach (e.g. [1, 4, 6]).

In the process of relating schemata and their operations to abstract datatypes we have defined the operations negation, disjunction and implication on abstract datatypes, which were previously not defined. Further work needs to be done to study the relationship of these new operations with the other operations on abstract datatypes, and how to integrate the new operations into proof calculi, like that of Hennicker, Wirsing and Bidoit [7]. Work in this direction has been done for the case of disjunction in Baumeister [2].

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