Time Series Analysis

Solutions to problems in Chapter 8

IMM

Solution 8.1

Question 1.

$$\begin{aligned} Y_t &= \sum_{p=0}^{k} a_p X_{t-p} = \left(\sum_{p=0}^{k} a_p B^p \right) X_t \Rightarrow \\ \mathcal{H}(\omega) &= \sum_{p=0}^{k} a_p e^{-i\omega p} \end{aligned}$$

The spectral density for $\{Y_t\}$ is found as

$$f_y(\omega) = G^2(\omega) f_x(\omega) = \mathcal{H}(\omega)\overline{\mathcal{H}(\omega)} f_x(\omega) \ ,$$

where

$$\begin{aligned} \mathcal{H}(\omega)\overline{\mathcal{H}(\omega)} &= \sum_{p=0}^{k} a_p e^{-i\omega p} \sum_{q=0}^{k} a_q e^{i\omega q} \\ &= \sum_{p=0}^{k} \sum_{q=0}^{k} a_p a_q e^{-i\omega q} e^{i\omega p} \\ &= \sum_{p=0}^{k} \sum_{q=0}^{k} a_p a_q e^{-i\omega(p-q)} \\ &= \sum_{p=0}^{k} \sum_{q=0}^{k} a_p a_q \cos(\omega(p-q)) + i \sum_{p=0}^{k} \sum_{q=0}^{k} a_p a_q \sin(\omega(p-q)) \\ &= \sum_{p=0}^{k} \sum_{q=0}^{k} a_p a_q \cos(\omega(p-q)) \end{aligned}$$

I.e.

$$\underline{\underline{f_y(\omega) = \sum_{p=0}^{k} \sum_{q=0}^{k} a_p a_q \cos(\omega(p-q)) f_x(\omega)}}$$

Question 2.

$$Y_t = \left(\frac{1}{k+1}\sum_{p=0}^{k} B^p\right) X_t = \frac{1}{k+1}\left(\frac{1-B^{k+1}}{1-B}\right) X_t$$

I.e.

$$\begin{aligned}
\mathcal{H}(\omega)\overline{\mathcal{H}(\omega)} &= \frac{1}{(k+1)^2}\left(\frac{1-e^{-i\omega(k+1)}}{1-e^{-i\omega}}\right)\left(\frac{1-e^{i\omega(k+1)}}{1-e^{i\omega}}\right) \\
&= \frac{1}{(k+1)^2}\frac{2-e^{-\omega(k+1)}-e^{i\omega(k+1)}}{2-e^{-i\omega}-e^{i\omega}} \\
&= \frac{1}{(k+1)^2}\frac{2(1-\cos(\omega(k+1)))}{2(1-\cos(\omega))} = \frac{1}{(k+1)^2}\frac{\sin^2(\omega(k+1)/2)}{\sin^2(\omega/2)}
\end{aligned}$$

The spectral density for $\{Y_t\}$ becomes

$$\underline{\underline{f_y(\omega) = \frac{1}{(k+1)^2}\frac{\sin^2(\omega(k+1)/2)}{\sin^2(\omega/2)} f_x(\omega)}}$$

Question 3.

The spectral density for $\{Z_t\}$ is

$$f_z(\omega) = \mathcal{H}(\omega)\overline{\mathcal{H}(\omega)} f_y(\omega) = \frac{1}{(k+1)^2}\frac{\sin^2(\omega(k+1)/2)}{\sin^2(\omega/2)} f_y(\omega)$$

I.e.

$$\underline{\underline{f_z(\omega) = \frac{1}{(k+1)^4}\frac{\sin^4(\omega(k+1)/2)}{\sin^4(\omega/2)} f_x(\omega)}}$$

Question 4.

Given

$$\begin{aligned}
Z_t &= 0.5Y_t + 0.5Y_{t-1} \\
Y_t &= 0.5X_t + 0.5X_{t-1}
\end{aligned}$$

The expression for $\{Z_t\}$ described by values of $\{X_t\}$ is

$$Z_t = 0.25X_t + 0.5X_{t-1} + 0.25X_{t-2}$$

The impulse response function is found by sending a pulse (i.e. a one) through the system.

$$h_k = \begin{cases} 0 & k < 0 \\ 0.25 & k = 1 \\ 0.5 & k = 2 \\ 0.25 & k = 3 \\ 0 & k > 2 \end{cases}$$

Question 5.

The frequency response function is

$$\mathcal{H}(\omega) = \sum_k h_k e^{i\omega k} = \underline{\underline{0.25 + 0.5e^{-i\omega} + 0.25e^{-i2\omega}}}$$

The amplitude (or gain) is determined as

$$\begin{aligned}
G^2(\omega) =& \mathcal{H}(\omega)\overline{\mathcal{H}(\omega)} \\
=& (0.25 + 0.5e^{-i\omega} + 0.25e^{-i2\omega})(0.25 + 0.5e^{i\omega} + 0.25e^{i2\omega}) \\
=& \frac{1}{16} + \frac{1}{8}e^{i\omega} + \frac{1}{16}e^{i2\omega} + \frac{1}{8}e^{-i\omega} + \frac{1}{4} + \frac{1}{8}e^{i\omega} + \frac{1}{16}e^{-i2\omega} + \frac{1}{8}e^{-i\omega} + \frac{1}{16} \\
=& \frac{3}{8} + \frac{1}{2}\cos(\omega) + \frac{1}{8}\cos(2\omega) \\
=& \frac{3}{8} + \frac{1}{2}\cos(\omega) + \frac{1}{8}(2\cos^2(\omega) - 1) \\
=& \left(\frac{1}{2}\right)^2 (1 + \cos(\omega))^2 \Rightarrow \\
G(\omega) =& 0.5(1 + \cos(\omega))
\end{aligned}$$

The phase is

$$\begin{aligned}
\Phi(\omega) = & \arg(\mathcal{H}(\omega)) = \arctan\left(\frac{-0.25\sin(\omega) - 0.25\sin^2(\omega)}{0.25 + 0.25\cos(\omega) + 0.25\cos^2(\omega)}\right) \\
= & \arctan\left(\frac{-0.25\sin(\omega) - 0.25\sin^2(\omega)}{0.25 + 0.25\cos(\omega) + 0.25(2\cos^2(\omega) - 1)}\right) \\
= & \arctan\left(\frac{-\sin(1 - \cos(\omega))}{\cos(\omega)(1 - \cos(\omega))}\right) \\
= & \arctan\left(\frac{\sin(-\omega)}{\cos(-\omega)}\right) = -\omega
\end{aligned}$$

The amplitude or phase could also be calculated more easily by applying that

$$\begin{aligned}
\mathcal{H}(\omega) = & (0.25e^{-i\omega} + 0.5 + 0.25e^{i\omega})e^{-i\omega} = \mathcal{H}_1(\omega)\mathcal{H}_2(\omega) \Rightarrow \\
G(\omega) = & G_1(\omega)G_2(\omega) = 0.25(1.\cos(\omega)) \cdot 1 = \underline{\underline{0.5(1.\cos(\omega))}} \\
\Phi(\omega) = & \underline{\underline{\Phi_1(\omega)\Phi_2(\omega) = 0 + (-\omega) = -\omega}}
\end{aligned}$$

I.e. the composed filter is seen as a symmetric filter ($h_{-1} = 0.25, h_0 = 0.5, h_1 = 0.25$) coupled to a time delay (with $\mathcal{H}(\omega) = e^{-i\omega}$). The amplitude and phase is plotted in Figure 1. From the amplitude plot it is clear that the composed filter is a low-pass filter.

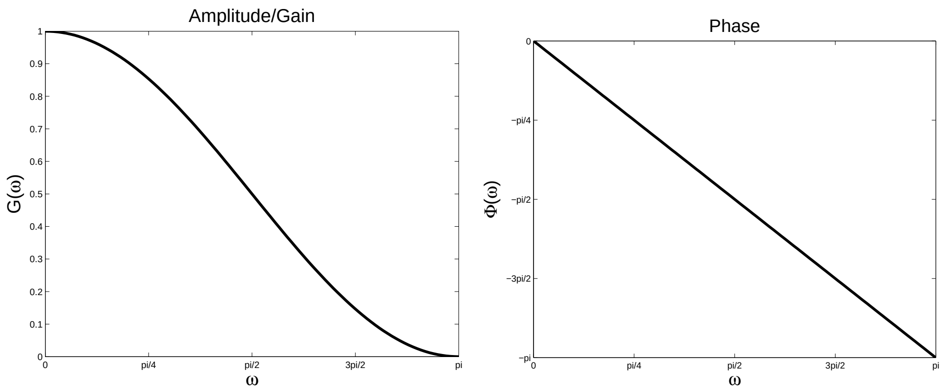

Figure 1: The amplitude (left) and phase (right) of the composed filter.

Question 6.

If the composed filter is changed to a filter with the following impulse response function

$$h_k = \begin{cases} 0 & k < -1 \\ 0.25 & k = -1 \\ 0.5 & k = 0 \\ 0.25 & k = 1 \\ 0 & k > 1 \end{cases}$$

the amplitude will be maintained but a phase shift is avoided ($\Phi(\omega) = 0$).

Solution 8.2

Question 1. The following intervention model is considered

$$Y_t - 200 = \frac{100B}{1-0.9B} I_t + \frac{1+0.3B}{1-0.9B} \epsilon_t$$

It is assumed that the effect of previous campaigns is negligible.
We introduce

$$X_t = Y_t - 200$$

A prediction of for instance $X_{8|7}$ require information about ϵ_7. This can be found by setting e.g. $\epsilon_4 = 0$ and from here determining the one-step prediction and prediction error. First the intervention model is rewritten to

$$\begin{aligned} X_t - 0.9X_{t-1} =& \epsilon_t + 0.3\epsilon_{t-1} \Rightarrow \\ X_{t+1} - 0.9X_t =& \epsilon_{t+1} + 0.3\epsilon_t \end{aligned}$$

I.e.

$$\hat{X}_{t+1|t} = 0.9X_t + 0.3\epsilon_t$$

And the one-step predictions are

$$\begin{aligned} \hat{X}_{5|4} =& 0.9X_4 + 0.3 \cdot 0 = 0.9 \cdot 8 = 7.2 \Rightarrow \\ \epsilon_5 =& X_5 - \hat{X}_{5|4} = 15 - 7.2 = 7.8 \\ \hat{X}_{6|5} =& 0.9 \cdot 15 + 0.3 \cdot 7.8 = 15.8 \Rightarrow \\ \epsilon_6 =& X_6 - \hat{X}_{6|5} = -4 - 15.8 = -19.8 \\ \hat{X}_{7|6} =& 0.9 \cdot (-4) - 0.3 \cdot 19.8 = -9.54 \Rightarrow \\ \epsilon_7 =& X_7 - \hat{X}_{7|6} = 7 + 9.54 = 16.5 \end{aligned}$$

Thus, the expected sales in week 8,9,10 and 11 are

$$\begin{aligned} \hat{X}_{8|7} &= 0.9 \cdot 7 + 0.3 \cdot 16.5 = 11.3 \Rightarrow \underline{\underline{\hat{Y}_{8|7} = 211.3}} \\ \hat{X}_{9|7} &= 0.9\hat{X}_{8|7} = 0.9 \cdot 11.3 = 10.2 \Rightarrow \underline{\underline{\hat{Y}_{9|7} = 210.2}} \\ \hat{X}_{10|7} &= 0.9\hat{X}_{9|7} = 0.9 \cdot 10.2 = 9.2 \Rightarrow \underline{\underline{\hat{Y}_{10|7} = 209.2}} \\ \hat{X}_{11|7} &= 0.9\hat{X}_{10|7} = 0.9 \cdot 9.2 = 8.3 \Rightarrow \underline{\underline{\hat{Y}_{11|7} = 208.3}} \end{aligned}$$

Question 2.

A sales campaign is undertaken in week 8, i.e. $I_8 = 1$. The expected sales in week 8 is as estimated in *Question 1*. For week 9-11 we get the following estimates

$$\begin{aligned}
&X_9 - 0.9X_8 = 100 \cdot 1 + \epsilon_9 + 0.3\epsilon_8 \Rightarrow \\
&\hat{X}_{9|7} = 0.9\hat{X}_{8|7} + 100 = 110.2 \Rightarrow \underline{\underline{\hat{Y}_{9|7} = 310.2}} \\
&X_{10} - 0.9X_9 = \epsilon_{10} + 0.3\epsilon_9 \Rightarrow \\
&\hat{X}_{10|7} = 0.9\hat{X}_{9|7} = 99.2 \Rightarrow \underline{\underline{\hat{Y}_{10|7} = 299.2}} \\
&X_{11} - 0.9X_{10} = \epsilon_{11} + 0.3\epsilon_{10} \Rightarrow \\
&\hat{X}_{11|7} = 0.9\hat{X}_{11|7} = 89.3 \Rightarrow \underline{\underline{\hat{Y}_{10|7} = 289.3}}
\end{aligned}$$

In order to determine a 95% confidence interval for the future sales we wish to rewrite the process such that we obtain the Ψ-weights.

$$X_t = \frac{100B}{1 - 0.9B} I_t + N_t$$

As $\mathrm{V}[I_t] = 0$ only N_t contribute to the prediction error, and the Ψ-weights are found by rewriting N_t into MA-form.

$$\begin{aligned}
N_t =& (1 + 0.3B)(1 + 0.9B + 0.9^2B^2 + 0.9^3B^3 + ...)\epsilon_t \\
=& (1 + 1.2B + 1.08B^2 + 0.97B^2 + ...)\epsilon_t \\
=& (1 + \Psi_1 B + \Psi_2 B^2 + \Psi_3 B^3 + ...)\epsilon_t
\end{aligned}$$

An approximate 95% confidence interval is obtained as

$$\hat{X}_{t+\ell|t} \pm 2 \cdot \sqrt{(1 + \Psi_1^2 + ... + \Psi_{\ell-1}^2)}\sigma_\epsilon$$

I.e.

$$\begin{aligned}
\text{uge 8:} \quad & 211.3 \pm 40.0 \quad \text{units/week} \\
\text{uge 9:} \quad & 310.2 \pm 62.5 \quad \text{units/week} \\
\text{uge 10:} \quad & 299.2 \pm 76.0 \quad \text{units/week} \\
\text{uge 11:} \quad & 289.3 \pm 85.3 \quad \text{units/week}
\end{aligned}$$