# Time Series Analysis

Solutions to problems in Chapter 4

IMM

# Solution 4.1

*Question 1.*

The system is given by the difference equation

$$y_t - 1.2y_{t-1} + 0.61y_{t-2} = x_t - 0.8x_{t-1}$$

Applying the Z-transform leads to

$$\left(1 - 1.2z^{-1} + 0.61z^{-2}\right) Y(z) = \left(1 - 0.8z^{-1}\right) X(z)$$

I.e.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - 0.8z^{-1}}{1 - 1.2z^{-1} + 0.61z^{-2}}$$

Since the roots to $z^2(1-1.2z^{-1}+0.61z^{-2}) = 0$ are $z = 0.6 \pm 0.5i$ the complex conjugated poles are

$$\underline{\underline{\pi_{1,p} = 0.6 + 0.5i \ , \ \pi_{2,p} = 0.6 - 0.5i}}$$

The zero is

$$\underline{\underline{\pi_{1,n} = 0.8}}$$

Because $|\pi_{1,p}| = |\pi_{2,p}| = (0.6^2 + 0.5^2)^{\frac{1}{2}} \simeq 0.7810 < 1$ the system is stable according to theorem 4.9

*Question 2.*

The impulse response function can be found by sending a '1' (i.e. a pulse) through the system. The values are found as

$$\begin{aligned}
&y_0 = x_0 = 1 \text{ the impulse is sent into the system.}\\
&y_1 = -0.8x_0 + 1.2y_0 = -0.8 \cdot 1 + 1.2 \cdot 1 = 0.4\\
&y_2 = 1.2y_1 - 0.61y_0 = 1.2 \cdot 0.4 - 0.61 \cdot 1 = -0.13\\
&y_3 = 1.2y_2 - 0.61y_1 = -1.2 \cdot 0.13 - 0.61 \cdot 0.4 = -0.4\\
&y_4 = 1.2y_3 - 0.61y_2 = -1.2 \cdot 0.4 + 0.61 \cdot 0.13 = -0.4007\\
&y_5 = 1.2y_4 - 0.61y_3 = -1.2 \cdot 0.4007 + 0.61 \cdot 0.4 = -0.2368
\end{aligned}$$

| t | $< 0$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|
| $x_t = s_t$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| $y_t = h_t$ | 0 | 1 | 0.4 | -0.13 | -0.4 | -0.4007 | -0.2368 |

Table 1: Solution to question 2.

The values are also summarized in Tabel 1. Because of the complex poles further calculations will reveal that the impulse response function is a damped oscillation.

*Question 3.*

The frequency response funktion is the values of the transfer function evaluated on the unit circle $Z(\omega) = e^{i\omega}$, and using the transfer function $H(z)$ from question 1 the frequency response function is

$$\mathcal{H}(\omega) = H(e^{i\omega}) = \underline{\underline{\frac{1 - 0.8e^{-i\omega}}{1 - 1.2e^{-i\omega} + 0.61e^{-i2\omega}}}}$$

The amplitude function is

$$\begin{aligned} G(\omega) &= |\mathcal{H}(\omega)| = \left[H(e^{i\omega})H(e^{-i\omega})\right] \\ &= \left[\frac{(1 - 0.8e^{-i\omega})(1 - 0.8e^{i\omega})}{(1 - 1.2e^{-i\omega} + 0.61e^{-i2\omega})(1 - 1.2e^{i\omega} + 0.61e^{i2\omega})}\right]^{1/2} \\ &= \left[\frac{1.64 - 1.6\cos\omega}{2.8121 - 3.864\cos\omega + 1.22\cos 2\omega}\right]^{1/2} \end{aligned}$$

A sketch of $G(\omega)$ can be seen in figure 1 ($G(\omega)$ is symmetric around the y-axis). The function has a maximum a bit to the left of $\pi/4 = 0.24\pi$.

The poles in the transfer funciton (see question 1) can also be written as

$$0.6 \pm 0.5i \simeq 0.78^2 e^{-i0.22\pi}$$

I.e. there is a connection between the argument to the complex conjugated pair and the argument which result in the maximum of $G(\omega)$.

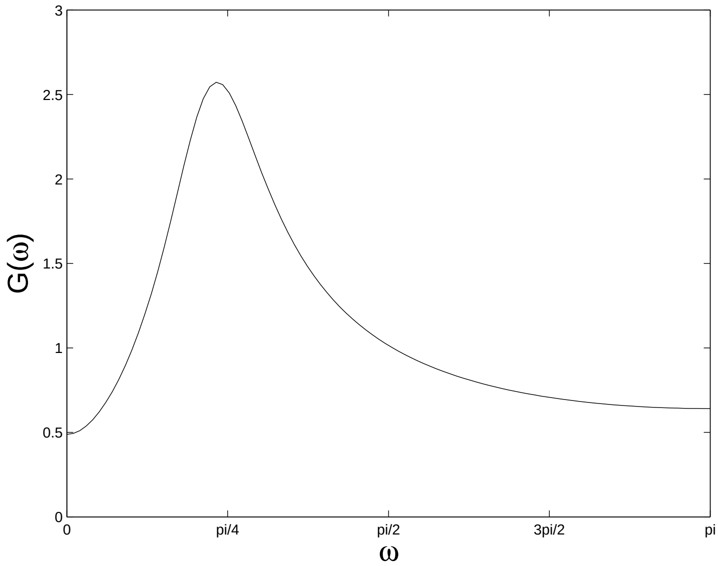


Figure 1: $G(\omega)$.

## Solution 4.2

*Question 1.*

The Laplace transform is applied in order to find the transfer function

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2 + 5s + 4} = \frac{1}{(s+1)(s+4)}$$

The poles of the system are $s_1 = -1$ and $s_2 = -4$. Since both poles are negative theorem 4.18 states that the system is stable.

*Question 2.*

By using (4.107) the frequency response function is

$$H(\omega) = H(i\omega) = \frac{1}{(i\omega + 1)(i\omega + 4)} = \frac{1}{4 - \omega^2 + 5i\omega}$$

*Question 3.*

The amplitude function is

$$G(\omega) = \left[H(i\omega)\overline{H(i\omega)}\right]^{1/2} = \left[\frac{1}{(\omega^2+1)(\omega^2+16)}\right]^{1/2} = \left[\frac{1}{\omega^4 + 17\omega^2 + 16}\right]^{1/2}$$

The amplitude function is shown in figure 2

By choosing the samplings frequency to $\omega_0 = 8\pi$ very small values of $G(\omega)$ will lie above the Nyquist frequency ($= \omega_0/2 = 4\pi$). The sampled signal in the frequency domain $Y_s(\omega)(-4\pi \leq \omega \leq 4\pi)$ will therefore resemble the continous signal (in the samme area) to a great extent.

The samplings frequency $\omega_0 = 8\pi$ correspond to a sampling time at $T = 2\pi/\omega_0 = 1/4$. A rule of thumb says that having a system dominated by real poles one should choose the sampling time of same order as the smallest time constant. The considered system have the time constants $r_1 = -1/s_1 = 1$ and $r_2 = -1/s_2 = 1/4$, i.e. the chosen sampling time is equal to the smallest time constant.

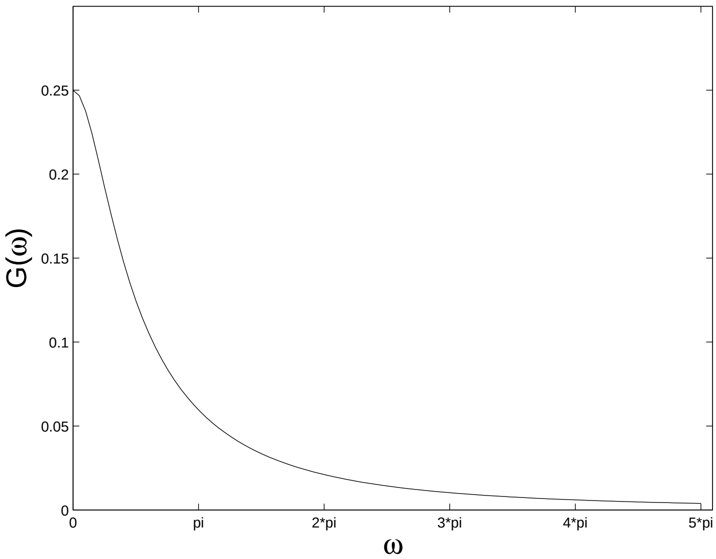


Figure 2: $G(\omega)$.

**Question 4.**

The poles in the sampled system are determined by the relation (from section 4.7)

$$z = e^{sT}$$

which leads to

$$z_1 = e^{-1 \cdot \frac{1}{4}} \simeq \underline{\underline{0.7768}}$$

$$z_2 = e^{-4 \cdot \frac{1}{4}} \simeq \underline{\underline{0.3678}}$$