Time Series Analysis

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Outline of todays lecture

Descriptions of (*deterministic*) linear systems.
Chapter 4: Linear Systems

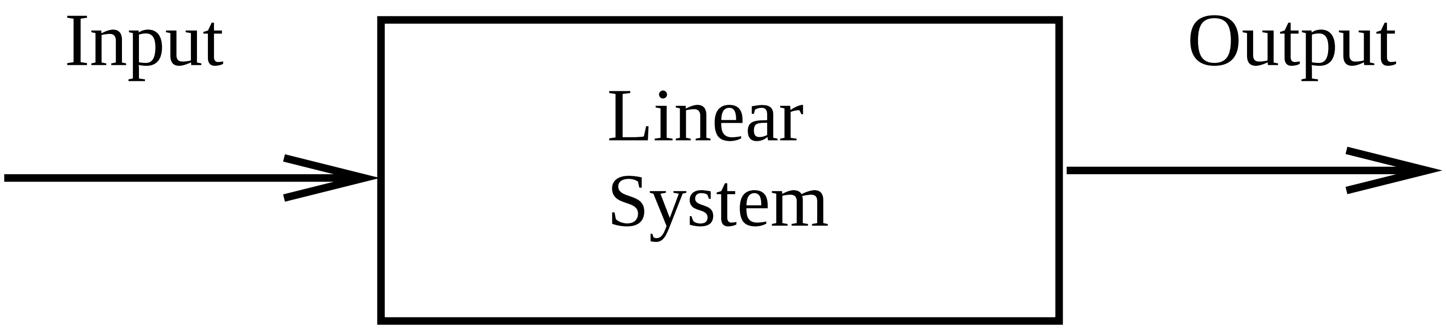
Linear (Input-Output) systems

- Linear systems, Chap. 4, except Sec. 4.7

Cursory material:

- Sec. 4.6

Linear Dynamic Systems



- We are going to study the case where we measure the input and the and the output to/from a system
- Here we will discuss some theory and descriptions for such systems
- Later on (in the next lecture) we will consider how we can *model* the system based on measurements of input and output.

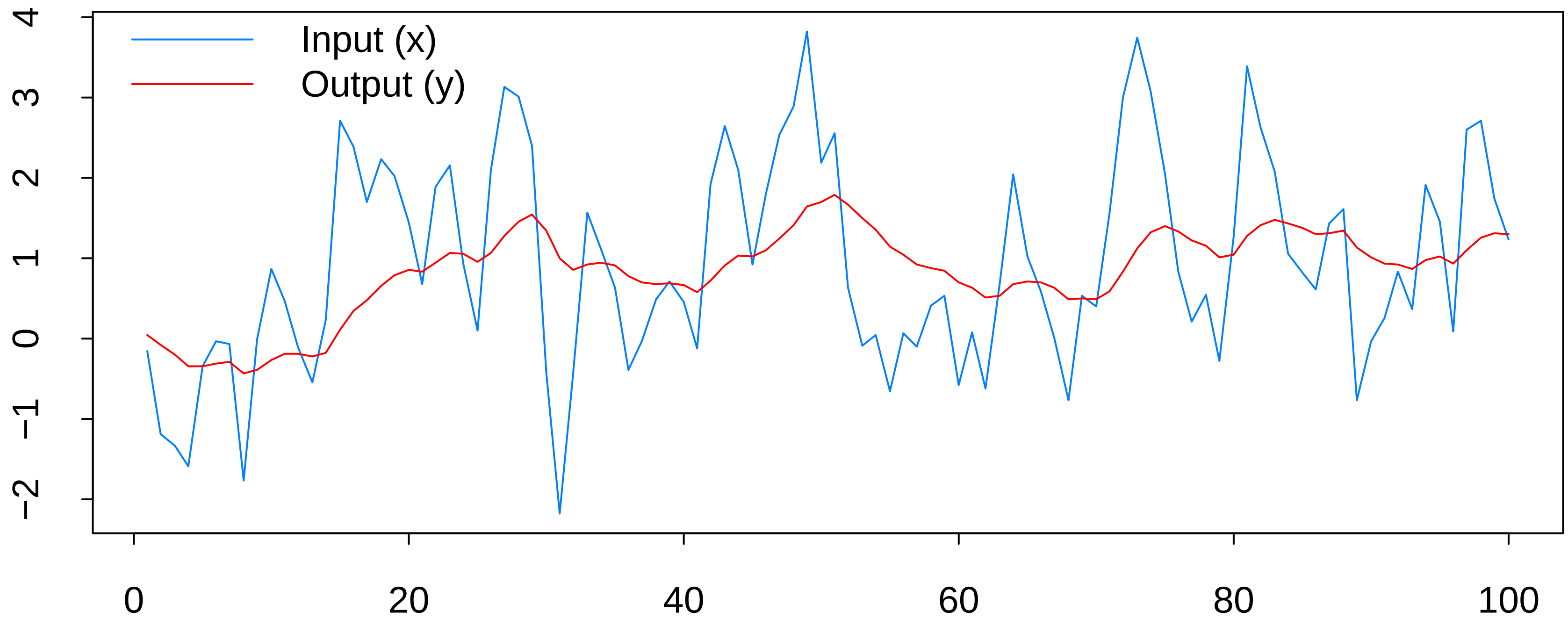
Dynamic response

- What would happen to the temperature inside a hollow, insulated, concrete block, which you
- place it in a controlled temperature environment,
- wait until everything is settled (all temperatures are equal), and then
- suddenly raise the temperature by $100^o C$ outside the block?

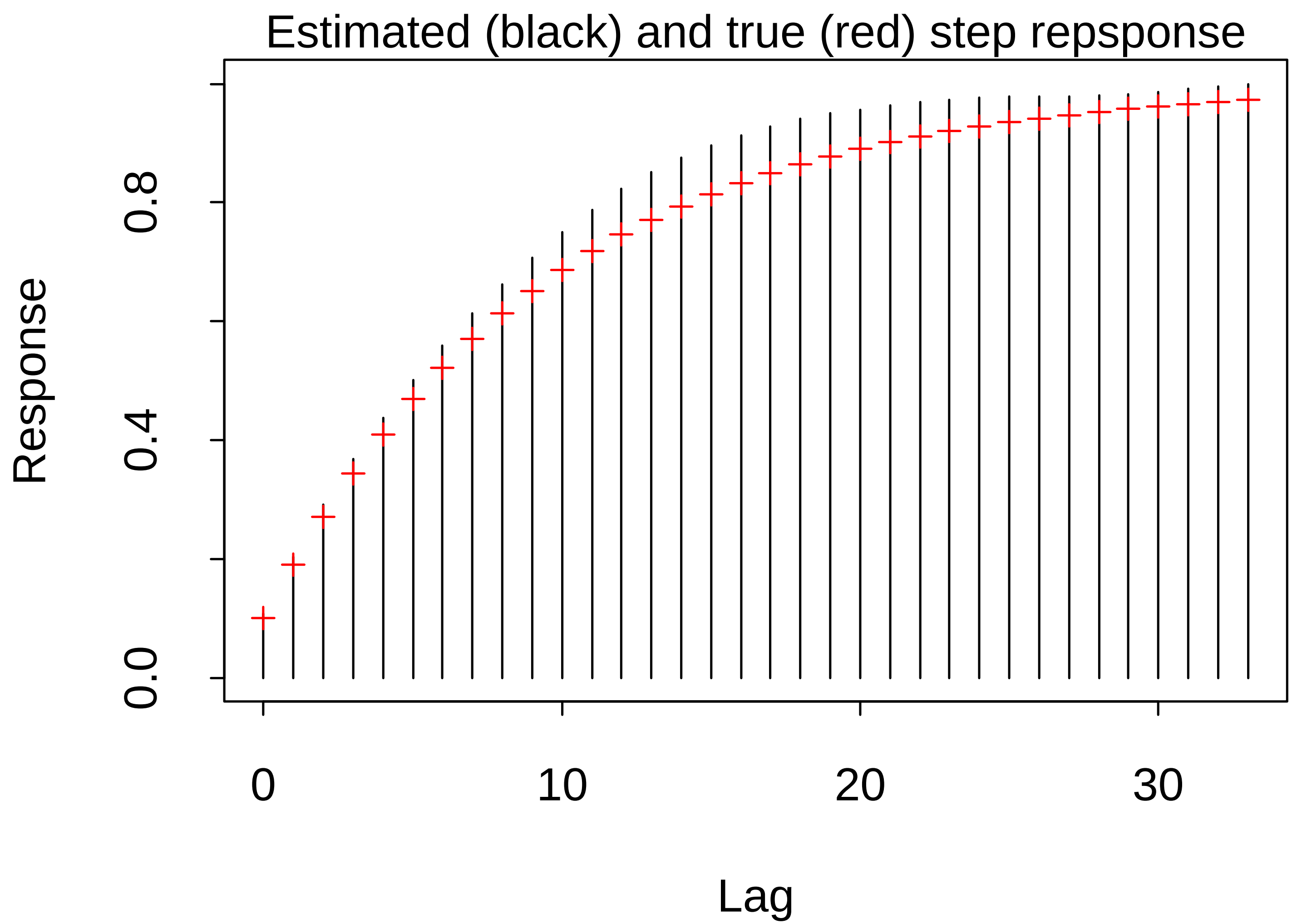
Sketch the temporal development of the temperature outside and inside the block

Dynamic response characteristics from data

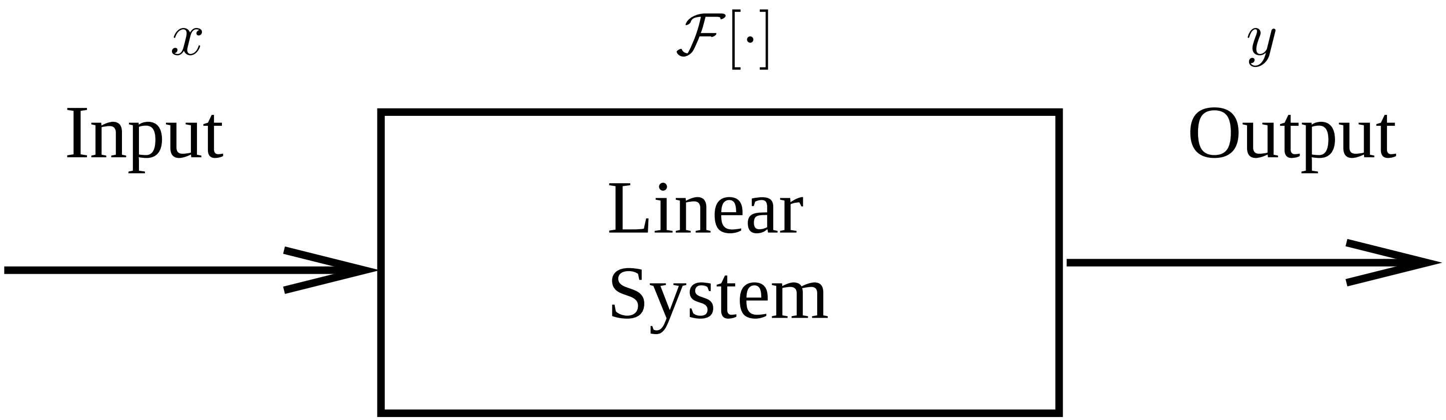
- An important aspect of what we aim at later on is to identify the characteristics of the dynamic response based on measurements of input and output signals



Dyn. response characteristics from data (cont’nd)



Linear Dynamic Systems – notation



$x(t)$	Differential eq., $h(u)$	$y(t)$
x_t	Difference eq., h_k, $h(B)$	y_t
$X(\omega)$	$\mathcal{H}(\omega)$	$Y(\omega)$
$X(z)$	$H(z)$	$Y(z)$
$(X(s)$	$H(s)$	$Y(s))$

Dynamic Systems – Some characteristics

Def. Linear system:

$$\mathcal{F}\left[\lambda_1 x_1(t) + \lambda_2 x_2(t)\right] = \lambda_1 \mathcal{F}\left[x_1(t)\right] + \lambda_2 \mathcal{F}\left[x_2(t)\right]$$

Def. Time invariant system:

$$y(t) = \mathcal{F}\left[x(t)\right] \Rightarrow y(t-\tau) = \mathcal{F}\left[x(t-\tau)\right]$$

Def. Stable system: A system is said to be *stable* if any constrained input implies a constrained output.

Def. Causal system: A systems is said to be *physically feasible* or *causal*, if the output at time t does not depend on future values of the input.

Example

- System: $y_t - a y_{t-1} = b x_t$
- Can be written: $y_t = b x_t + a y_{t-1} = b x_t + a(b x_{t-1} + a y_{t-2})$ or

$$y_t = b(x_t + a x_{t-1} + a^2 x_{t-2} + a^3 x_{t-3} + \ldots) = b \sum_{k=0}^{\infty} a^k x_{t-k}$$

- The system is seen to be *linear* and *time invariant*
- The *impulse response* is $h_k = b a^k, \; k \geq 0$ (0 otherwise) and the system is seen to be *causal*
- Since

$$\sum_{k=-\infty}^{\infty} |h_k| = \sum_{k=0}^{\infty} |b||a|^k = \begin{cases} |b|/(1-|a|) & ; \quad |a| < 1 \\ \infty & ; \quad |a| \geq 1 \end{cases}$$

the system is *stable* for $|a| < 1$ (stability does not depend on b)

Description in the time domain

For *linear time invariant systems*:

- Continuous time:

$$y(t) = \int_{-\infty}^{\infty} h(u)x(t-u)\,du \qquad (1)$$

- Discrete time:

$$y_t = \sum_{k=-\infty}^{\infty} h_k x_{t-k} \qquad (2)$$

- $h(u)$ or h_k is called the *impulse response*
- $S_k = \sum_{j=-\infty}^{k} h_j$ is called the *step response* (similar def. in continuous time)
- The impulse response can be determined by “sending a 1 trough the system”

Example: Calculation of the impulse response fct.

The impulse can be determined by 'sending a 1 trough the system'. Consider the linear, time invariant system

$$y_t - 0.8y_{t-1} = 2x_t - x_{t-1} \tag{3}$$

By putting $x = \delta$ we see that $y_k = h_k = 0$ for $k < 0$. For $k = 0$ we get

$$\begin{aligned} y_0 &= 0.8y_{-1} + 2\delta_0 - \delta_{-1} \\ &= 0.8 \times 0 + 2 \times 1 - 0 = 2 \end{aligned}$$

i.e. $h_0 = 2$.

Example - Cont.

Going on we get

$$
\begin{aligned}
y_1 &= 0.8y_0 + 2\delta_1 - \delta_0 = 0.8 \times 2 + 2 \times 0 - 1 = 0.6 \\
y_2 &= 0.8y_1 = 0.48 \\
&\vdots \\
y_k &= 0.8^{k-1}0.6 \ (k > 0)
\end{aligned}
$$

Hence, the impulse response function is

$$
h_k = \begin{cases} 0 & \text{for } k < 0 \\ 2 & \text{for } k = 0 \\ 0.8^{k-1}0.6 & \text{for } k > 0 \end{cases}
$$

which clearly represents a causal system. Furthermore, the system is stable since $\sum_0^\infty |h_k| = 2 + 0.6(1 + 0.8 + 0.8^2 + \cdots) = 5 < \infty$

Description in the frequency domain

- The *Fourier transform* is a way of representing a signal $y(t)$ or y_t by it's distribution over frequencies:

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)\mathrm{e}^{-i\omega t}\,dt \quad \text{or} \quad Y(\omega) = \sum_{t=-\infty}^{\infty} y_t \mathrm{e}^{-i\omega t}$$

- If the time unit is seconds, ω is the *angular frequency* in radians per second. In discrete time $-\pi \leq \omega < \pi$
- For a linear time invariant system it holds that

$$Y(\omega) = \mathcal{H}(\omega)X(\omega)$$

where $\mathcal{H}(\omega)$ is the Fourier transform of the impulse response function. $\mathcal{H}(\omega) = |\mathcal{H}(\omega)|\mathrm{e}^{i\arg\{\mathcal{H}(\omega)\}} = G(\omega)\mathrm{e}^{i\phi(\omega)}$

Description in the frequency domain (cont.)

- The function $\mathcal{H}(\omega)$ is called the *Frequency response function*, and it is the Fourier transformation of the impulse response function, ie.

$$\mathcal{H}(\omega) = \sum_{k=-\infty}^{\infty} h_k e^{-i\omega k} \quad (-\pi \le \omega < \pi) \tag{4}$$

- The frequency response function is complex. Thus, it is possible to split $\mathcal{H}(\omega)$ into a real and a complex part:

$$\mathcal{H}(\omega) = |\mathcal{H}(\omega)| e^{i \arg\{\mathcal{H}(\omega)\}} = G(\omega) e^{i\phi(\omega)} \tag{5}$$

where $G(\omega)$ is the *amplitude* (*amplitude function*) and $\phi(\omega)$ is the *phase* (*phase function*).

Single harmonic input

- Lets consider the *a single harmonic signal* as input:

$$x(t) = A\mathrm{e}^{i\omega t} = A\cos\omega t + iA\sin\omega t \tag{6}$$

- Then the output becomes also single harmonic, cf.:

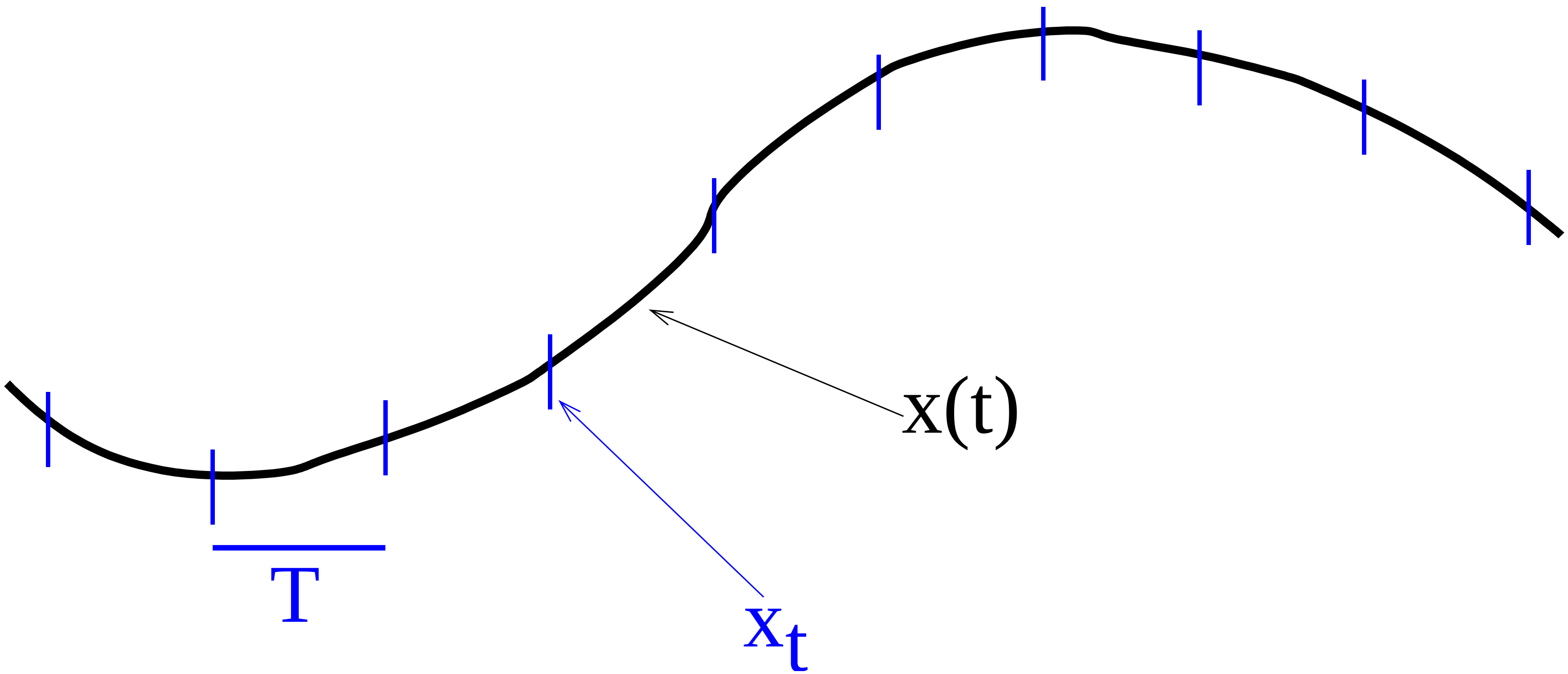
$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(u)x(t-u)\,du \\ &= \int_{-\infty}^{\infty} h(u)A\mathrm{e}^{i\omega(t-u)}\,du \\ &= A\mathrm{e}^{i\omega t}\int_{-\infty}^{\infty} h(u)\mathrm{e}^{-i\omega u}\,du \\ &= H(\omega)A\mathrm{e}^{i\omega t} = G(\omega)A\mathrm{e}^{i\left(\omega t+\phi(\omega)\right)} \end{aligned} \tag{7}$$

Single harmonic input (cont.)

- A *single harmonic* input to a linear, time invariant system will give an output having the *same frequency* ω. The amplitude of the output signal equals the amplitude of the input signal multiplied by $G(\omega)$. The change in phase from input to output is $\phi(\omega)$.

Sampling

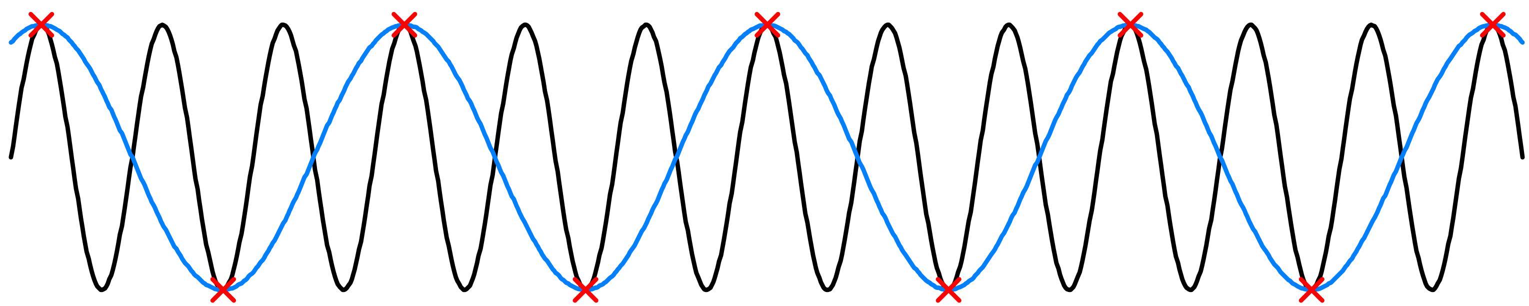
From continuous time to discrete time – what is lost?



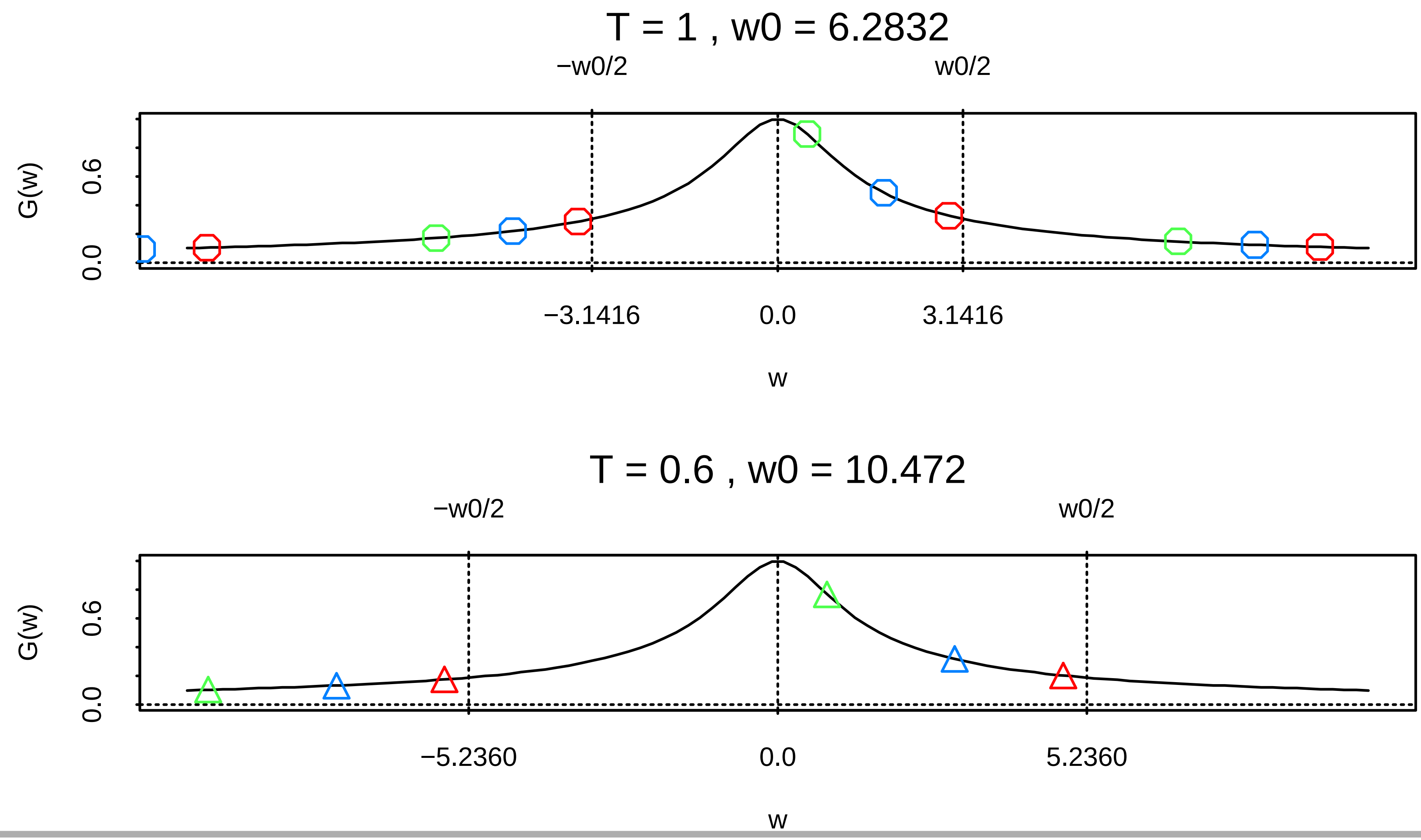
- T is the *sampling time*
- $\omega_0 = 2\pi/T$ is the *sampling frequency*

Sampling (cont'nd)

- If we work out the mathematical theory of sampling it turns out that the Fourier transform of the sampled signal $X_s(\omega)$ is composed by the Fourier transform of the original signal $X(\omega)$ at the correct frequency ω and at the frequencies $\omega \pm \omega_0$, $\omega \pm 2\omega_0$, $\omega \pm 3\omega_0$, ...
- If $X(\omega)$ is zero outside the interval $[-\omega_0/2, \omega_0/2] = [-\pi/T, \pi/T]$ then $X_s(\omega) = X(\omega)$
- If not the values outside the interval cannot be distinguished from values inside the interval (*aliasing*)



Sampling (cont'nd)



The z-transform

- A way to describe dynamical systems in discrete time

$$Z(\{x_t\}) = X(z) = \sum_{t=-\infty}^{\infty} x_t z^{-t} \qquad (z \text{ complex})$$

- The z-transform of a time delay: $Z(\{x_{t-\tau}\}) = z^{-\tau} X(z)$
- The *transfer function* of the system is called $H(z) = \sum_{t=-\infty}^{\infty} h_t z^{-t}$

$$y_t = \sum_{k=-\infty}^{\infty} h_k x_{t-k} \quad \Leftrightarrow \quad Y(z) = H(z)X(z)$$

- Relation to the *frequency response function*: $\mathcal{H}(\omega) = H(\mathrm{e}^{i\omega})$

Linear Difference Equation

$$
\begin{aligned}
y_t + a_1 y_{t-1} + \cdots + a_p y_{t-p} &= b_0 x_{t-\tau} + b_1 x_{t-\tau-1} + \cdots + b_q x_{t-\tau-q} \\
(1 + a_1 z^{-1} + \cdots + a_p z^{-p}) Y(z) &= z^{-\tau}(b_0 + b_1 z^{-1} + \cdots + b_q z^{-q}) X(z)
\end{aligned}
$$

Transfer function:

$$
\begin{aligned}
H(z) &= \frac{z^{-\tau}(b_0 + b_1 z^{-1} + \cdots + b_q z^{-q})}{(1 + a_1 z^{-1} + \cdots + a_p z^{-p})} \\
&= \frac{z^{-\tau}(1 - n_1 z^{-1})(1 - n_2 z^{-1}) \cdots (1 - n_q z^{-1}) b_0}{(1 - \lambda_1 z^{-1})(1 - \lambda_2 z^{-1}) \cdots (1 - \lambda_p z^{-1})}
\end{aligned}
$$

Where the roots $n_1, n_2, \ldots, n_q$ is called the *zeros of the system* and $\lambda_1, \lambda_2, \ldots, \lambda_p$ is called the *poles of the system*

The system is stable if all poles lie within the unit circle

Relation to the backshift operator

$$\begin{aligned} y_t + a_1 y_{t-1} + \cdots + a_p y_{t-p} &= b_0 x_{t-\tau} + b_1 x_{t-\tau-1} + \cdots + b_q x_{t-\tau-q} \\ (1 + a_1 z^{-1} + \cdots + a_p z^{-p}) Y(z) &= z^{-\tau}(b_0 + b_1 z^{-1} + \cdots + b_q z^{-q}) X(z) \\ (1 + a_1 B^1 + \cdots + a_p B^p) y_t &= B^\tau (b_0 + b_1 B^1 + \cdots + b_q B^q) x_t \\ \varphi(B) y_t &= \omega(B) B^\tau x_t \end{aligned}$$

The output can be written:

$$y_t = \varphi^{-1}(B)\omega(B)B^\tau x_t = h(B)x_t = \left[\sum_{i=0}^{\infty} h_i B^i\right] x_t = \sum_{i=0}^{\infty} h_i x_{t-i}$$

$h(B)$ is *also* called the *transfer function*. Using $h(B)$ the system is assumed to be causal; compare with $H(z) = \sum_{t=-\infty}^{\infty} h_t z^{-t}$

Estimating the impulse response

- The poles and zeros characterize the impulse response (Appendix A and Chapter 8)
- If we can estimate the impulse response from recordings of input an output we can get information that allows us to *suggest a structure for the transfer function*